# Productivity Driven Quality: Theory and US Calibration

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**Abstract**

How does relative productivity affect relative quality? In the United States, how much variation in services-goods relative quality can be accounted for with changes in relative productivity alone? In the model, labor is used for both quantity production and quality innovation. A change in productivity induces a reallocation of labor, leading to quality variation. This study finds that relative productivity's effect on relative quality depends on how substitutable the products are (substitution parameter) and how easy it is to improve quality (innovation parameter). Specifically, productivity and quality have a negative correlation for low-range substitutability and a positive correlation for medium-range substitutability, where the medium range depends on the innovation parameter. Applying the model to the US services-goods economy from 1970-2006 using NIPA data, the study finds that productivity plays a quantitatively important role in quality innovation. In addition, (relative) productivity and quality of the services sector have a negative correlation.

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## 1 Introduction

At disaggregate levels, we can actually observe quality changes, e.g. better cars or new bank services coming out each year. However, at different aggregate levels, quality changes are not directly observable. Nguyen (2007b) proposes a simple method which relies on relative prices to infer relative quality changes.

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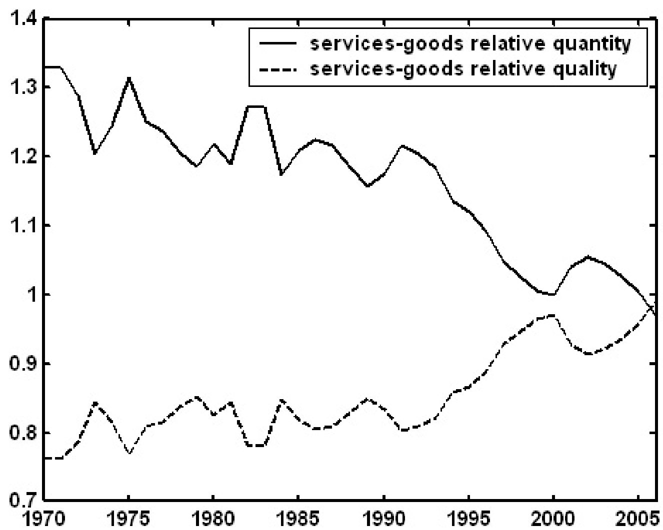
The idea behind the inference method is that, in equilibrium, relative prices of different products should be equal to ratios of marginal utility, and marginal utility depends on quantity and quality of consumption. Thus a relative price should be a function of relative quantity and relative quality. As relative price and relative quantity can be observed, we can back out relative quality. Applying this inference method to the goods and services sectors of the US from 1970-2006, we see a positive trend in quality of services relative to goods. In addition, relative quantity and relative quality of services are negatively correlated (Figure 1). This negative correlation is striking in face of the fact that Nguyen (2007b) treats relative quantity and relative quality as exogenous variations. In other words, the result suggests relative quality of services is not simply random and is endogenously linked to relative productivity which drives relative quantity.

Figure 1: Services-goods relative quantity & quality in US, 1970-2006




*Note*: the time series are extracted from Nguyen (2007b) where raw data comes from NIPA tables; relative quantity is normalized to unit in 2000, while relative quality comes directly from the inference method; the mirror-imaged variations of the two time series are necessary to explain a smooth relative price path (Figure 3).

Motivated by the negative correlation above, we address two questions. First, how does relative productivity affect relative quality? Second, in the United States, how much variation in relative quality of services can be accounted for with changes in relative productivity alone? Answers to these questions are interesting because this is an area that we know little about. In addition, the answers are potentially important for policies related to factor markets, e.g. labor and capital income taxes. Specifically, if productivity plays a big role in quality innovation, or, alternatively, quality innovation is an important by-product of productivity variation, we should base our optimal tax policies on a model of the economy where quality responds to productivity rather than a model where quality responses are not incorporated.

Our current model has two important features. First, labor is used for both quantity production and quality innovation. As quantity and quality can be substitutable in consumption, a productivity change induces a reallocation of labor between different activities, leading to quality variation, which is called productivity-driven quality. This is a reality rather than just a hypothesis. For example, teachers can impose limits on class sizes to improve teaching quality. Researchers may refrain from carrying out simultaneously too many projects to work more on each project. Thus, a productivity change may eventually vary output quality. Second, to minimize monetary effects on measurements of real variables, we divide the economy into two sectors and define objects in relative terms, i.e. services versus goods. This modeling choice helps enrich theoretical examinations and apply the model to available aggregate data.

Here are the major findings. First, via both analytical and computational approaches, we find that (relative) productivity's effect on (relative) quality depends on two key parameters, which govern how substitutable the products are (substitution parameter) and how easy it is to improve quality (innovation parameter). Specifically, productivity and quality have a negative correlation for low-range substitutability and a positive correlation for medium-range substitutability, where the upper bound of the medium range negatively depends on the innovation parameter. This result holds for both models with and without capital accumulation. Second, we extend the model so that aggregate quality plays a role in quantity production to see if the standard growth accounting method remains valid. The results show no big differences between the baseline and extended models. The reason is that individual quality indices move in opposite directions, keeping aggregate quality stable. This implies the baseline model is rich enough for empirical application. Third, when applying the baseline model to the US services-goods economy from 1970-2006, we see that changes in productivity alone can account for as much as one half in growth of relative quality of the services sector. We do not expect to account for all variation in quality with changes in productivity alone because general quality innovation is also due to all other possible sources like randomly developed ideas for quality improvement and races for market power. Fourth, we pin down some key parameters for the US. The parameter estimates imply a negative correlation between relative productivity and quality of the services sector.

The significance of productivity-driven quality calls for more policy attentions at this type of quality response. For example, a tax policy designed based on a model economy which does not incorporate productivity-driven quality can lead to suboptimal outcomes in reality. This result does not mean that we have to build productivity-driven quality in all macroeconomic models where productivity is the ultimate source of variation. However, this type of quality response can be included in sensitivity analyses or thought experiments to see its missing is benign or not.

To put our study in a perspective, quality innovation essentially serves as a propagation mechanism of productivity changes in growth and real business-cycle (RBC) models. Since the seminal work by Kydland & Prescott (1982), there has been a vastly developing literature using aggregate productivity to explain variations in quantity and variety of output and consumption. What missing is a line of investigation on how productivity affects quality. As far as we know, our model is among the first to explore this third branch. One possible explanation for the thin literature on productivity-driven quality at the aggregate level is the prominent lack of good measurements of quality changes as well as statistics on resources used for quality innovation. The statistics community has been attempting to correct biases in price data to have augmented measures of output which also reflect quality changes. This statistical attempt is far from perfect (Nguyen 2007b). The bigger problem is that, even though lumping quantity and quality together can be useful for welfare measurements, it is not beneficial for understanding the true behavior of the economy in allocating resources and designing optimal tax policies.

There are two lines of literature that embrace two dimensions of quality changes. One is the "quality ladder" literature in which varieties are fixed while quality of products is evolving with a constant step in any period. The other is the "variety growth" literature in which quality of a single product is fixed and the number of products is changing, e.g. Grossman & Helpman (1991) and Aghion & Howitt (1992), respectively. In the current study, we fix the number of products and allow quality to be endogenously determined. We deviate from the "quality ladder" literature in two aspects: (i) markets are perfectly rather than imperfectly competitive; and (ii) quality level and innovation steps are continuous in the sense of "quality escalator" (Nguyen 2007a). We choose "quality escalator" over "quality ladder" and "variety growth" for several reasons. First, "quality escalator" allows for continuous quality ratios, richer sets of innovation decisions, and a better fit in empirical applications than "quality ladder". Second, "variety growth" can be equivalently represented by "quality escalator". For example, total utility $\int_0^\theta u(x)di$ can be replaced by single utility $\theta u(x)$. With the statistical practice, though not perfect, that separates quantity growth from price changes and reduces the large product space into a small discrete set, "quality escalator" is more endowed than "variety growth".

It is also worthwhile discussing the difference between taste shock and quality change. Taste shock is a random change in valuation of the same products. Quality change is a variation in the nature of the product, either endogenous or exogenous, that alters agents' valuation. At the aggregate level and with low frequency data, it is hard to interpret taste shock as a synchronized event happening to all agents. Meanwhile, aggregate quality change can be automatically achieved via competition and imitation. We assume that a time length of one year is enough for sectoral quality synchronization.

The remaining of the study is structured as follows. Section 2 sets up the model. Section 3 characterizes the relationship between productivity and quality. Section 4 applies the model to the US economy. Finally, Section 5 concludes.

## 2 An Economy of Two Sectors

In this section, we look at primitives of the environment, a competitive equilibrium notion, and an extension of the baseline model.

### 2.1 Baseline Preference and Technology

The economy has one representative consumer and two competitive commodity sectors. Sectors 1 and 2 produce perfectly divisible products $a$ and $b$, respectively. The consumer is endowed with one unit of labor which is perfectly mobile between the sectors. The numeraire is good $a$ at time zero, i.e. $p_{a0} = 1$. The representative agent has the life-time expected utility

$$E_0 \sum_{t=0}^{\infty} \lambda^t u\left(a_t, b_t; \alpha_t, \beta_t\right),\ 0 < \lambda < 1 \tag{1}$$

$$u\left(a_t, b_t; \alpha_t, \beta_t\right) = \frac{1}{1-\sigma}\left\{\left(\left[\left(\alpha_t a_t\right)^{\theta} + \left(\beta_t b_t\right)^{\theta}\right]^{1/\theta}\right)^{1-\sigma} - 1\right\}, \tag{2}$$

where $\sigma > 0$ and $\theta \leq 1$ are correspondingly for intertemporal and contemporary substitution. With $\theta$ as the *substitution parameter*, the elasticity of substitution is $-1/(1-\theta)$. As $\theta$ ranges from $-\infty$ to 1, absolute value of the elasticity of substitution ranges from 0 to $\infty$. The degree of substitution is greater than unit if $\theta \in (0, 1]$ and smaller than unit if $\theta < 0$.

The representative agent faces budget, labor, capital, and technology constraints, $\forall t \geq 0$, as follows

$$p_{at}\left(a_t + x_{at}\right) + p_{bt}\left(b_t + x_{bt}\right) = w_t\left(l_{at} + l_{bt}\right) + r_t\left(k_{at} + k_{bt}\right) \tag{3}$$

$$l_{at} + l_{bt} + n_{at} + n_{bt} = 1 \tag{4}$$

$$k_{at} + k_{bt} = k_t \tag{5}$$

$$k_{t+1} = (1-\delta)\, k_t + \phi x_{at}^{\nu} x_{bt}^{1-\nu} \tag{6}$$

$$\alpha_t = \left(1 - \eta_\alpha\right) \alpha_{t-1} + \left(n_{at}\right)^{\xi},\ 0 < \eta_\alpha < 1 \tag{7}$$

$$\beta_t = \left(1 - \eta_\beta\right) \beta_{t-1} + \left(n_{bt}\right)^{\xi},\ 0 < \eta_\beta < 1, \tag{8}$$

where positive $(a_t, b_t)$ and positive $(\alpha_t, \beta_t)$ denote quantity and quality, respectively; $(x_{at}, x_{bt})$ are nonnegative investments towards the aggregate capital stock $k_{t+1}$; $(k_{at}, k_{bt})$ are capital services from the current stock $k_t$; $(l_{at}, l_{bt})$ are labor used for quantity production, while $(n_{at}, n_{bt})$ are for quality innovation; $w_t$ is the wage rate for $(l_{at}, l_{bt})$; $r_t$ is the capital rental rate; $\nu \in (0, 1)$ is the *contribution factor*; $\phi > 0$ is the *scale factor*; $\delta$ is the capital depreciation rate; and $\xi \in (0, 1)$ is the *innovation parameter*. Note that, under perfect competition, profits will be zero and play no part in the budget constraint.

Thus, besides two consumption products, there is one investment good used only for production. The law of motion for aggregate capital stock in (6) has the mixing feature used by, for example, Kehoe & Ruhl (2005). In addition, the agent uses "home production" to acquire know-how for improving product quality. Specifically, quality indices of $a$ and $b$ respectively have the laws of motion in (7) and (8). "Home production" simply means that it is the agent rather than firms who accumulates know-how which will then be transferred into product quality. For example, a teacher can revise some syllabus at his or her own will, either at home or at school.

On the quantity production side, each sector is represented by a competitive firm. The firms have Cobb-Douglas quantity production functions

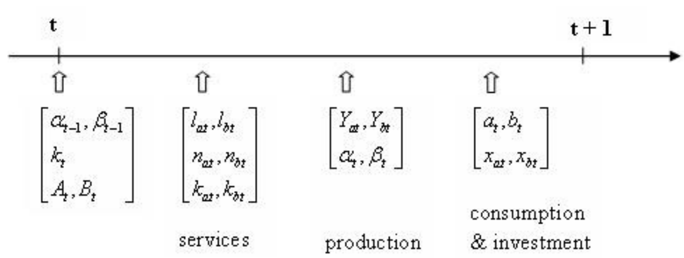
$$Y_{at} = A_t K_{at}^{\mu} L_{at}^{1-\mu}, \ 0 < \mu < 1 \tag{9}$$

$$Y_{bt} = B_t K_{bt}^{\gamma} L_{bt}^{1-\gamma}, \ 0 < \gamma < 1, \tag{10}$$

where $(K_{at}, K_{bt})$ and $(L_{at}, L_{bt})$ are demanded capital and labor, respectively; and strictly positive $(A_t, B_t)$ are sectoral total factor productivity (TFP) which may follow some deterministic or stochastic processes, and have a common aggregate variation component. In our model, productivity $(A_t, B_t)$ is the sole and ultimate exogenous source of variations.

Before explicitly laying out the timing of the economy, we have several notes. First, the effective consumption quantity is a product of physical quantity and quality index, neither of which can be zero. Second, the representative agent will not use all of labor endowment for quantity production, because he or she also wants to spend some time on quality innovation. If the agent does not spend any time on innovation, product quality depreciates as the result of forgetfulness. Third, for simplicity, we rule out the possibility of reversing investment good into consumption products. Fourth, capital services are purely quantitative, e.g. number of machines. Fifth, sectoral productivity only has direct effects on quantity production.

Figure 2: Timing of the economy




Let $\omega_t = \{\alpha_{t-1}, \beta_{t-1}, k_t, A_t, B_t\}$ be the information set at the beginning of period $t$, where $(\alpha_{t-1}, \beta_{t-1})$ denote quality of the products produced at time $t-1$; $k_t$ is the capital stock; and $(A_t, B_t)$ are sectoral productivity shocks. Timing of the economy in period $t \geq 0$ is as follows (Figure 2). First, information set $\omega_t$ is observed. Second, the representative agent decides on factor uses $\{(l_{at}, l_{bt}); (n_{at}, n_{bt}); (k_{at}, k_{bt})\}$. Third, the representative firms produce their output quantities $(Y_{at}, Y_{bt})$ with new quality levels $(\alpha_t, \beta_t)$ supplied by the agent.

Fourth, given the factor incomes and quality levels, the representative agent decides on how much to consume and invest $\{(a_t, b_t); (x_{at}, x_{bt})\}$ to maximize the continuing expected life-time utility. This sequence makes it clear that the agent may want to spend some efforts on quality innovation. In essence, all the actions can happen simultaneously at point $t$.

## 2.2 A Perfectly Competitive Equilibrium

Besides primitives in preference and technology, we need rules for the interactions on markets: a perfectly competitive equilibrium. Before defining the equilibrium, we formalize the utility and profit maximization problems.

Let the extended information set in period $t$ be $\widehat{\omega}_t = \omega_t \cup \{(\alpha_t, \beta_t)\}$; and $C_{1t} = \{(l_{at}, l_{bt}); (n_{at}, n_{bt}); (k_{at}, k_{bt})\}$, $C_{2t} = \{(a_t, b_t); (x_{at}, x_{bt})\}$ be the decisions. The representative agent maximizes the life-time expected utility following the dynamic programming problem

$$V(\omega_t) = \max_{C_{1t}, C_{2t}} \{u(a_t, b_t; \alpha_t, \beta_t) + \lambda E_{\widehat{\omega}_t} V(\omega_{t+1})\} \tag{11}$$

subject to the constraints in (3)-(5), the law of motion for the capital stock in (6), and the evolution of quality indices in (7)-(8).

On the production side, the representative firms solve the static profit maximization problems for every period $t$

$$\max_{\{L_{at}, K_{at}\}} \left\{p_{at} A_t K_{at}^{\mu} L_{at}^{1-\mu} - w_t L_{at} - r_t K_{at}\right\} \tag{12}$$

$$\max_{\{L_{bt}, K_{bt}\}} \left\{p_{bt} B_t K_{bt}^{\gamma} L_{bt}^{1-\gamma} - w_t L_{bt} - r_t K_{bt}\right\}, \tag{13}$$

with the necessary and sufficient conditions

$$w_t = (1-\mu)\frac{p_{at} Y_{at}}{L_{at}} = (1-\gamma)\frac{p_{bt} Y_{bt}}{L_{bt}} \tag{14}$$

$$r_t = \mu \frac{p_{at} Y_{at}}{K_{at}} = \gamma \frac{p_{bt} Y_{bt}}{K_{bt}}. \tag{15}$$

**Definition 2.1.** *A perfectly competitive equilibrium of the economy is the set of policy functions* $\{(l_{at}, l_{bt}); (n_{at}, n_{bt}); (k_{at}, k_{bt}); (a_t, b_t); (x_{at}, x_{bt})\}_{t=0}^{\infty}$ *for the representative agent,* $\{(L_{at}, L_{bt}); (K_{at}, K_{bt})\}_{t=0}^{\infty}$ *for the representative firms, prices* $\{(p_{at}, p_{bt}); (w_t, r_t)\}_{t=0}^{\infty}$*, and the associated value function* $V(\omega_t)$ *such that*

*(i) given prices, policy functions are solutions to the dynamic programming problem (11) and the profit maximization problems (12)-(13);*

*(ii) given policy functions, prices clear all markets*

$$a_t + x_{at} = Y_{at} \tag{16}$$

$$b_t + x_{bt} = Y_{bt} \tag{17}$$

$$L_{at} = l_{at} \tag{18}$$

$$L_{bt} = l_{bt} \tag{19}$$

$$K_{at} = k_{at} \tag{20}$$

$$K_{bt} = k_{bt}. \tag{21}$$

In equilibrium, the representative firms make zero profits, leaving the utility maximization problem (11) valid. Thus, the definition of a perfectly competitive equilibrium completes our baseline economy.

### 2.3 Hypothesis of Augmented Capital

In the baseline environment, quality does not have any effect on quantity production. We relax this assumption in a simple extension. Let $\overline{q}_t$ be the average quality of the investment good, and $\overline{q}_t$ follows the evolution

$$\overline{q}_{t+1} = (1 - S_q)\,\overline{q}_t + S_q \alpha_t^{\nu} \beta_t^{1-\nu},\ S_q \in (0, 1)\,, \tag{22}$$

where $S_q$ is the weight for the addition of new quality mix. The hypothesis of augmented capital says that effective capital services embed capital quality. Specifically, the modified sectoral production functions have the forms

$$Y_{at} = A_t\,(\overline{q}_t K_{at})^{\mu}\, L_{at}^{1-\mu} \tag{23}$$

$$Y_{bt} = B_t\,(\overline{q}_t K_{bt})^{\gamma}\, L_{bt}^{1-\gamma}. \tag{24}$$

In reality, we do not often directly observe capital quality. However, we can observe output quantity, capital stock, and labor. Thus, a direct application of the standard growth accounting method on (23), for example, will generate a productivity measure of $A_t\,(\overline{q}_t)^{\mu}$ rather than of $A_t$. The question is: how does our introduction of capital quality alter the TFP estimation in the baseline model? The answer to this question may guide us in choosing the right model in empirical studies, i.e. with or without capital quality.

## 3 Productivity-Quality Causation

This section is devoted to theoretical examinations, especially the causation from productivity to quality. We first characterize the baseline economy and then deal with the augmented capital hypothesis. In the baseline economy, the equilibrium allocation is Pareto optimal. This Pareto optimal allocation can be implemented through a competitive equilibrium with some price system. Thus, to solve for the perfectly competitive equilibrium, we follow a two-step algorithm: (i) find the equilibrium allocation with a social planner problem; and (ii) given the equilibrium allocation, derive the equilibrium prices.

**Definition 3.1.** *The social planner finds the functions* $\{C_{1t}, C_{2t}\}_{t=0}^{\infty}$*, with* $C_{1t} = \{(l_{at}, l_{bt}); (n_{at}, n_{bt}); (k_{at}, k_{bt})\}$ *and* $C_{2t} = \{(a_t, b_t); (x_{at}, x_{bt})\}$*, as solutions to the dynamic programming problem*

$$V(\omega_t) = \max_{C_{1t}, C_{2t}} \{u(a_t, b_t; \alpha_t, \beta_t) + \lambda E_{\widehat{\omega}_t} V(\omega_{t+1})\} \tag{25}$$

*subject to the constraints*

$$l_{at} + l_{bt} + n_{at} + n_{bt} = 1 \tag{26}$$

$$k_{at} + k_{bt} = k_t \tag{27}$$

$$a_t + x_{at} = A_t k_{at}^{\mu} l_{at}^{1-\mu} \tag{28}$$

$$b_t + x_{bt} = B_t k_{bt}^{\gamma} l_{bt}^{1-\gamma}, \tag{29}$$

*the law of motion for aggregate capital stock in (6), and evolution of product quality in (7)-(8).*

In Definition 3.1, the maximization problem has a strictly concave objective function with a convex compact constraint set. That means the social planner solution uniquely exists. Given the optimal allocation, the equilibrium wage and interest rates are derived from (14)-(15). In addition, we can pin down the product price ratio for every period $t$ based on the condition that marginal utility-price ratios should be equated between the two sectors

$$p_{bat} = \left(\frac{\beta_t}{\alpha_t}\right)^{\theta} \left(\frac{b_t}{a_t}\right)^{\theta-1}, \tag{30}$$

where $p_{bat} = p_{bt}/p_{at}$, and the budget share for $b$ follows the identity

$$S_{bt} = \frac{p_{bt} b_t}{p_{at} a_t + p_{bt} b_t}. \tag{31}$$

Necessary and sufficient conditions for the optimal decisions in Definition 3.1 constitute a complicated system, i.e. (A.5)-(A.14).

So far, we have had conditions for the social planner solution and corresponding decentralized equilibrium in a representative agent model. As $\xi \in (0, 1)$, the representative agent will spend positive amounts of time on quality innovation even though he or she does not receive compensation for accumulating know-how from firms. In our model, the innovation incentive comes from the desire for more utility in consumption. Starting from zero innovation efforts, the representative agent can always do better by spending infinitesimal amount of time because marginal gains are very large, i.e. slopes of the innovation functions in (7) and (8) are infinite at zero.

One question arises. What happens to innovation efforts if there are a continuum of agents, which is a reality, rather than a representative agent? Our concern is that the agents may free ride on the efforts of one another, and hence spend no time on quality innovation. To stop this behavior and come back to the representative agent equilibrium, we need to restrict our considerations to symmetric equilibria. The argument runs as follows. First, there is no symmetric equilibrium in which all agents make zero innovation efforts. The reason is, as mentioned before, the stakes at zero efforts are so high that someone will deviate and accumulate some know-how. Second, for well-behaved preference and technology primitives, there exists a set of general equilibria including a symmetric one. All we need to do next is to select this symmetric equilibrium and come back to the representative agent case. Finally, the symmetry restriction is valid because we are currently interested in the average behavior of the economy.

## 3.1 Comparative Statics

There are several motivations behind our study of comparative statics in some steady state indexed by sectoral productivity $(A, B)$. First, given $(A, B)$, we can analytically solve for the steady-state equilibrium including allocations and a supporting price system which is unique up to some scale (Appendix A.2). Second, in the steady state, we can perturb productivity to learn about the productivity-quality causation and other equilibrium behaviors of interest. Third, we can identify the parametric subspace which is relevant for the steady states and later refine that for the equilibrium dynamics. In overall, this is a good starting point for exploring our economy.

**Lemma 3.2.** *Given $\Lambda = \nu(\mu - 1) + (1 - \nu)(\gamma - 1) \neq 0$, $\theta \neq \frac{1}{1+\xi}$, and some $A$, changes in relative productivity $B/A$ have qualitative effects on relative consumption, quality, price, and budget share, respectively, as follows*

$$\begin{aligned} sign\left[\partial\left(b/a\right)/\partial\left(B/A\right)\right] &= sign\left[\left(1-\theta\xi\right)\left(\theta\xi+\theta-1\right)\Lambda\right] && (32)\\ sign\left[\partial\left(\beta/\alpha\right)/\partial\left(B/A\right)\right] &= sign\left[\theta\left(\theta\xi+\theta-1\right)\Lambda\right] && (33)\\ sign\left[\partial p_{ba}/\partial\left(B/A\right)\right] &= sign\left[\Lambda\right] && (34)\\ sign\left[\partial S_b/\partial\left(B/A\right)\right] &= sign\left[\theta\left(\theta\xi+\theta-1\right)\Lambda\right]. && (35) \end{aligned}$$

**Proof.** Appendix A.2. ∎

The restrictions in Lemma 3.2 deserve some explanations. First, $\Lambda = (\mu - 1)\nu + (\gamma - 1)(1 - \nu) \neq 0$. Recall that $(1 - \mu)$ and $(1 - \gamma)$, under perfect competition, are the labor shares respectively in sector 1 and sector 2, and $\nu$ is the contribution share in accumulating the aggregate capital stock. Thus $|\Lambda|$ is an average labor share. The restriction means that labor should play some part in the economy. Generically, $\mu \in (0, 1)$, $\gamma \in (0, 1)$, $\nu \in (0, 1)$, and hence $\Lambda \in (-1, 0)$. Second, $\theta \neq \frac{1}{1+\xi}$, or the elasticity of substitution $-1/(1-\theta)$ cannot be $-(1+\frac{1}{\xi})$. This condition guarantees that the quality ratio $\beta/\alpha$ is well defined. Recall that $\xi \in (0, 1)$ is the curvature in quality evolution functions (7)-(8). In words, $\xi$ positively governs the degree of easiness in quality innovation, and hence plays a key role in the dynamics of quality indices. When $\theta = \frac{1}{1+\xi}$, quality indices are canceled out in the system of FOCs, specifically (A.51), and become indeterminate. Besides this restriction, generically, $\theta < 1$, i.e. products are not linearly substitutable, and hence $\theta\xi < 1$. In addition, from (33) and (35), it is more interesting to have $\theta \neq 0$. Third, sectoral productivity $A$ is fixed in Lemma 3.2. The reason is that, except for $\nu = 0.5$, the left-hand-side objects in (32)-(35) cannot be expressed as functions of only $B/A$ without extra $A$ or $B$ terms. In the symmetry case, i.e. $\nu = 0.5$, all those objects can be defined as closed-form functions of relative productivity $B/A$.

In combination, $\Lambda < 0$, $\theta < 1$, $\theta \neq 0$, and $\theta \neq \frac{1}{1+\xi}$. Conditional on $\xi$, let $\theta < 0$ be *little substitutability*; $\theta \in (0, \frac{1}{1+\xi})$ be *medium substitutability*; and $\theta \in (\frac{1}{1+\xi}, 1)$ be *large substitutability*. There are some observations based on Lemma 3.2 as follows (note that every object is in relative terms).

**Proposition 3.3.** *Given some $A$, $\Lambda < 0$, $\theta < 1$, $\theta \neq 0$, $\theta \neq \frac{1}{1+\xi}$, and conditional on $\xi \in (0, 1)$, in the steady states: (i) productivity affects consumption positively for little-medium substitutability, and negatively for large substitutability; (ii) productivity affects quality positively for medium substitutability, and negatively for little or large substitutability; (iii) productivity always has negative effects on relative price; and (iv) productivity affects budget share in the same way as quality. These results hold for all $A$ if $\nu = 0.5$.*

Thus, key links in the economy qualitatively depend on the substitution parameter $\theta$, conditional on $\xi$. We have some insights for the results in Proposition 3.3. The initial effect of a positive productivity shock in sector 2, i.e. a surge in $B$, is a potential increase in supply of $b$. As the supply of $b$ increases, marginal utility of this product decreases, pressing the relative price $p_{ba}$ down. Part (iii) says this negative relative price effect prevails no matter what happens next, independent of the substitution pattern. That is a surge in $B$ would finally make $b$'s marginal value decrease relatively to that of $a$. To understand parts (i), (ii), and (iv) we consider three cases. For little substitutability, the utility function with respect to effective consumption of either $a$ or $b$ is quite concave. That means the agent wants to keep a close balance between $\alpha \times a$ and $\beta \times b$. As quantity $b$ increases, the agent transfer some capital and labor from sector $b$ to produce more $a$ and improve quality $\alpha$, making $\beta/\alpha$ decrease. In this case, negative price effect dominates positive quantity effect, making the budget share for $b$ decrease. For medium substitutability, the agent does not need to keep a close balance between effective consumptions of $a$ and $b$. In addition, as effective consumption is a product between quantity and quality, the agent benefits the most by improving both quantity and quality of largely one sector. As quantity $b$ increases, it is optimal for the agent to improve quality $\beta$. In this case, positive response in supply of $b$ overwhelms reduction in $p_{ba}$, leading to a larger $S_b$. For large substitutability, again, the agent only needs to work mostly on one sector. As products are now very easy to be substituted, the freed-up resources after a surge in $B$ will be spent mostly on quantity and quality of product $a$. This behavior differs from the case of medium substitutability and does not exist in a model without quality innovation. As the agent demands more $a$ relative to $b$, and with a decrease in $p_{ba}$, the budget share for $b$ decreases.

When there is no capital accumulation, effects of relative productivity on relative quality have the same patterns as in Proposition 3.3 (Appendix A.6). Clearly, $\xi$ and $\theta$ play important roles in what to produce, what to consume, and what to improve upon. In empirical studies, these parameters should be fitted to data. Within some ball around the steady state, we expect that those results in (32)-(35) would hold for equilibrium dynamics.

## 3.2 Equilibrium Dynamics

With a numerical exercise, we look at equilibrium dynamics of the economy in the neighborhood of some steady state, i.e. an RBC model. For simplicity, the equilibrium is approximated by linear laws of motion of endogenous variables. The exercise will generate correlation coefficients shedding light on the economic relations of interest. We use the parameter values in Table 1 for simulations. The processes governing $(A_t, B_t)$ will be specified shortly.

Table 1: Baseline parameter values in simulations

| description | symbol | range | value |
|---|---|---|---|
| curvature of CES | $\theta$ | $(-\infty, 1]$ | $-3; 0.3; 0.7$ |
| time discount factor | $\lambda$ | $(0, 1)$ | 0.95 |
| curvature of CRRA | $\sigma$ | $(0, \infty)$ | 2.0 |
| capital depreciation rate | $\delta$ | $(0, 1)$ | 0.05 |
| investment contribution factor | $\nu$ | $(0, 1)$ | 0.45 |
| investment scale factor | $\phi$ | $(0, \infty)$ | 1.0 |
| capital share in $a$ production | $\mu$ | $(0, 1)$ | 0.4 |
| capital share in $b$ production | $\gamma$ | $(0, 1)$ | 0.35 |
| depreciation of $\alpha$ | $\eta_\alpha$ | $(0, 1)$ | 0.1 |
| depreciation of $\beta$ | $\eta_\beta$ | $(0, 1)$ | 0.12 |
| quality innovation parameter | $\xi$ | $(0, 1)$ | 2/3 |

There are reasons behind the parameter choice. First, as noted earlier, conditional on $\xi$, the substitution parameter plays an important role in the relations between variables. We define $\xi = 2/3$, which means the threshold $\frac{1}{1+\xi} = 0.6$, and examine three specific values of $\theta$ corresponding to little, medium, and large substitutability. Specifically, the corresponding $\theta$ values are $-3$, 0.3, and 0.7. Second, the subjective discount rate is assumed to be 5.2 percent, and the discount factor is $\lambda = 0.95$. Third, the constant rate of risk aversion is $\sigma = 2$. In fact, $\sigma$ does not play any role in comparative statics. However, it critically governs intertemporal decisions in equilibrium dynamics. Fourth, the capital depreciation rate is assumed to be 5 percent per year. Fifth, it is more interesting to look at sectoral asymmetry, i.e. we have different capital shares and quality depreciation rates for the two sectors. Finally, for simplicity, $\phi = 1$.

The time path $\{A_t, B_t\}_{t=1}^T$ is generated according to a VAR model

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.7 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} A_{t-1} \\ B_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{at} \\ \varepsilon_{bt} \end{bmatrix}, \tag{36}$$

where $A_1 = 1$; $B_1 = 1$; and $\varepsilon_t = [\varepsilon_{at}\ \varepsilon_{bt}]' \overset{i.i.d.}{\sim} N(\mathbf{0}, \Sigma)$ with

$$\mathbf{\Sigma} = \begin{bmatrix} 2.04 \times 10^{-4} & 0 \\ 0 & 4.6875 \times 10^{-4} \end{bmatrix}. \tag{37}$$

Effectively, $A_t \sim N(1, 0.02^2)$, $B_t \sim N(1, 0.025^2)$. For simplicity, we do not allow $A_t$ and $B_t$ to be correlated. This restriction is not important for our analyses.

As some key variables can be observed in reality, model correlation patterns can be matched with data counterparts for clues on the range of the parameters $\xi$ and $\theta$. Recall that decisions are functions of the information set $\omega$. We first log-linearize the system of FOCs, and then impose the laws of motion $z_t = \Gamma z_{t-1} + \Psi s_t$, where $z_t$ includes key endogenous variables and $s_t$ contains the exogenous productivity shocks (Appendix A.3). In the log-linearization transformation, the variables $\{z_t, s_t\}$ are interpreted as percentage deviations from the nonstochastic steady state counterparts. As soon as transformed data are simulated, we can back out the original variables and generate correlation coefficients. The process that generates $s_t$ is derived from (36) and (37). The feedback and feedforward matrices $\Gamma$ and $\Psi$ are solved with the method of undetermined coefficients by Christiano (2002).

In this exercise, eigenvalues of $\Gamma$ for $\theta = \{-3, 0.3\}$ are strictly less than unit, keeping the dynamic systems stable. However, the dynamic system becomes unstable for $\theta = 0.7$. In fact, for $\theta \in (0.6, 1)$, the system is not stable most of the time; and if it is stable, quality indices may have negative values. For $\theta < 0.6$, e.g. $\theta = 0.59$, the system is stable (Appendix A.3). In general, by varying $\xi$ and $\theta$, we see that the system is always stable for $\theta < \frac{1}{1+\xi}$, and it becomes chaotic most of the time for $\theta > \frac{1}{1+\xi}$. In words, too much substitution between products $a$ and $b$ does not produce stable and sensible dynamics. With no friction in factor mobility, productivity variations may cause too much turnover of labor and capital between the two product sectors.

Table 2: Dynamics: correlation patterns

| line | description | $\theta < 0$ | $\theta \in (0, \frac{1}{1+\xi})$ |
|---|---|---|---|
| 1. | $B/A$ & $Y_b/Y_a$ | + | + |
| 2. | $B/A$ & $b/a$ | + | + |
| 3. | $B/A$ & $\beta/\alpha$ | − | + |
| 4. | $B/A$ & $p_{ba}$ | − | − |
| 5. | $B/A$ & $S_b$ | − | + |
| 6. | $B/A$ & $(l_b + n_b)/(l_a + n_a)$ | − | + |
| 7. | $B/A$ & $l_b/l_a$ | − | + |
| 8. | $B/A$ & $n_b/n_a$ | − | + |
| 9. | $B/A$ & $k_b/k$ | − | + |
| 10. | $Y_{bt}/Y_{at}$ & $b_t/a_t$ | + | + |
| 11. | $Y_{bt}/Y_{at}$ & $p_{bat}$ | − | − |
| 12. | $b_t/a_t$ & $p_{bat}$ | − | − |
| 13. | $\beta_t/\alpha_t$ & $p_{bat}$ | + | − |

*Note*: $\theta$ for substitution; $\xi$ for innovation.

Thus, we further impose that $\theta < \frac{1}{1+\xi}$ for $\xi \in (0, 1)$. What do we have with this restriction? The correlation patterns turn out to be consistent with the results in Proposition 3.3. We summarize the simulation results for the equilibrium dynamics in Table 2. First, as expected, the correlation between $B/A$ and $p_{ba}$ does not depend on $\theta$. Second, the correlation between $Y_b/Y_a$

and $b/a$ is always positive, e.g. if relative output is higher, the corresponding relative consumption also increases. In addition, $Y_b/Y_a$ and $b/a$ have the same correlation patterns with $B/A$, or $p_{ba}$, conditional on $\theta$. This implies that, in empirical studies, we only need data on either $Y_b/Y_a$ or $b/a$. Third, the correlation between relative productivity and quality lines up with Proposition 3.3. Specifically, a positive relative productivity shock decreases relative quality for little substitutability, and the reverse holds for medium substitutability. This is also the case in the relationship between $B/A$ and $S_b$. Fourth, for little substitutability or $\theta < 0$, relative quality and relative price have positive correlation. This result differs from that in Nguyen (2007b) even though the two models arrive at quite similar relative price functions as in (30). The reason for the difference lies in the nature of the relationship between relative quality and relative price. Nguyen (2007b) looks at a direct relationship between relative quality and relative price, in which the former is exogenous and causes the latter. Thus, for $\theta < 0$, relative quality and relative price have a negative correlation. In the current study, both relative quality and relative price are endogenously driven by the same exogenous productivity shocks, and have positive correlation when $\theta$ is negative. In addition, the difference also happens for $\theta \in (0, 0.5)$. In reality, if quality has some exogenous components which dominate the effects driven by productivity shocks, we may again see the results in Nguyen (2007b).

Lines 6-9 of Table 2 also support our predictions about how the agent reallocates labor and capital when facing a positive shock in relative productivity $B/A$. For little substitutability, to keep a close balance between effective consumptions, the agent transfers some labor and capital from sector 2 to sector 1, reducing relative quality $\beta/\alpha$. Note that relative output $Y_b/Y_a$ still rises for a dominant increase in $B/A$. For medium substitutability, there is not much difference between the two products. Thus, an increase in $B/A$ stimulates a diversion of resources from sector 1 to sector 2, raising $\beta/\alpha$.

### 3.3 Does Capital Quality Matter?

The extended model, in which capital quality plays some role in production functions, is presented in Appendix A.4. We examine the equilibrium dynamics of this model. In fact, the introduction of capital quality does not really affect growth accounting because in all cases $\overline{q}_t$ virtually does not change over time. In other words, for example, the time series $A_t$ and $A_t(\overline{q}_t)^\mu$ are nearly the same after rescaling. The direct reason is that no matter how the ratio $\beta_t/\alpha_t$ evolves, individual quality indices tend to move in opposite directions. e.g. when $\beta_t$ increases, $\alpha_t$ decreases. The intuition is that if changing the ratio $\beta_t/\alpha_t$ is one instrument for utility maximization, then moving innovation labor between $\alpha_t$ and $\beta_t$ is a direct way. As individual quality indices move in opposite directions, aggregate quality, which is a quality mixture (22), should not change much over time. We only consider this mixture specification purely for simplicity.

We have characterized the equilibrium dynamics in an RBC model. How about a growth model? Our established results remain valid. We embed considerations of a growth model in the upcoming US application.

# 4 Application to US Services-Goods Economy

Based on the previous discussion, we choose the baseline model to be fitted to the services and goods sectors of the US economy. Sector 1 produces goods ($a$), and sector 2 offers services ($b$). The ultimate exogenous driving force in the economy is productivity evolution in the two sectors. The object of our main interest is services-goods relative quality driven by productivity changes. There are three specific tasks. First is finding the parameter values so that the numerical model can generate certain moments in the data trends. Second is perturbing the productivity shocks around the trends to see if the model can mimic the patterns in data time paths and infer endogenous quality innovation. Third is comparing productivity-driven quality innovation with total quality innovation, which is based on Nguyen (2007b). Before carrying out the tasks, we examine the data.

## 4.1 Data Description

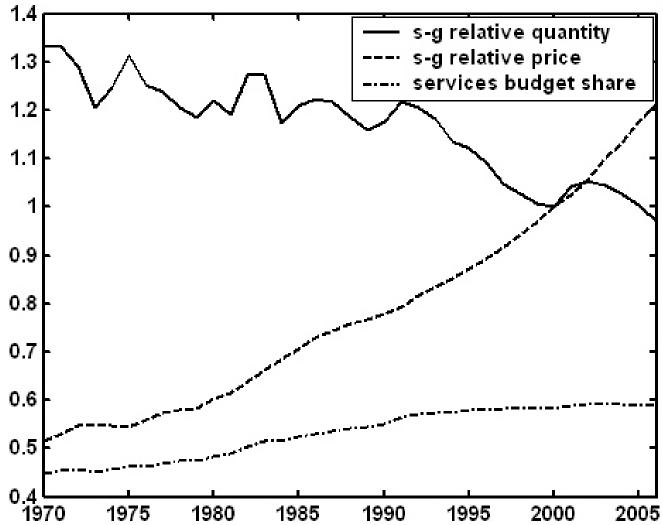
Our US data come from the national income and product accounts (NIPA) by the Bureau of Economic Analysis (BEA). All the data are publicly available and can be readily downloaded from the BEA's Web site. Postwar data are available from 1946 to 2006. However, only the 1970-2006 data are used. The horizon restriction is based on the fact that time series in 1970-2006 have clear trends which are useful to the upcoming moment matching exercise.

Classifications of goods and services follow the definitions of NIPA tables. The broad components of goods industries are agriculture, forestry, and fisheries; mining; and manufacturing. The services industries are transportation and public utilities; wholesale trade; retail trade and automobile services; finance, insurance, and real estate; different services; and government services. We omit residential and non-residential structures because they are composed of mixed quality levels and can render services for a very long period of time.

The original time series and their corresponding source NIPA tables are: (i) goods quantity index $Y_{at}$ (1.2.3); (ii) goods price index $P_{at}$ (1.2.4); (iii) services quantity index $Y_{bt}$ (1.2.3); (iv) services price index $P_{bt}$ (1.2.4); and (v) budget share for services $S_{bt}$ (1.5.5). Besides these key time series, there are other data pieces which will be specified later. All the quantity and price time series are first normalized so that their indices equal unit in the year 2000. The services-goods relative price is constructed as $P_{bt}/P_{at}$. Individual prices may have a lot of noises like inflation and taxes. However, the price ratio is supposed to largely bear relative quantity and quality information (Nguyen 2007b). By the same token, the budget share for services should evolve mostly under relative quantity and quality effects. It is noted that, reliable measures of labor allocation in quantity production and quality innovation are not available. This means we cannot use the standard growth accounting exercise to estimate sectoral productivity evolutions.

There are some key data features. First, goods and services quantities have upward trends with relatively stable growth rates (Figures 4 and 5). Second,

Figure 3: US services-goods economy, 1970-2006




*Source*: NIPA tables (BEA).

quantity of goods grows more quickly than that of services, leading to a downward trend in the services-goods relative quantity time path (Figure 3). Third, the services-goods relative price and the budget share for services are increasing over time. As predicted by the model, relative quantity and price are always negatively correlated. The opposite trends in the services relative quantity and budget share suggest that goods and services have little substitutability, i.e. $\theta < 0$, which seems intuitive for this product dichotomy.

In combination, the tasks have to rely on a limited data set. In addition, the economy we are trying to match is not the original one, but a normalized version of that. This normalization may cause inconsistencies between different objects. Next, we will discuss how to deal with these problems.

## 4.2 Parameter Values

The task is to numerically specify the baseline model. Among the parameters, $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$ are of our main interest because they critically govern the (relative) quality innovation process. In the literature, many studies rely on detrended data to estimate parameters. In this application, we do not rely on detrended data because the relative price and budget share concepts prevent us from doing so. In addition, we cannot apply GMM estimation methods based on the FOC system because productivity, detailed labor and capital uses, and quality indices are not directly observed. Our strategy is to first specify uncritical parameters and then use moment matching to pinpoint the four mentioned above. Values of the parameters are summarized in Table 3 followed by some explanatory details.

Table 3: Parameter values in US application

| description | symbol | value | source |
|---|---|---|---|
| curvature of CES | $\theta$ | $-7.0$ | matching |
| time discount factor | $\lambda$ | 0.97 | assigning |
| curvature of CRRA | $\sigma$ | 2.0 | assigning |
| capital depreciation rate | $\delta$ | 0.06 | assigning |
| investment contribution factor | $\nu$ | 0.42 | assigning |
| investment scale factor | $\phi$ | 1.0 | assigning |
| capital share in $a$ production | $\mu$ | 0.32 | assigning |
| capital share in $b$ production | $\gamma$ | 0.34 | assigning |
| depreciation of $\alpha$ | $\eta_\alpha$ | 0.0045 | matching |
| depreciation of $\beta$ | $\eta_\beta$ | 0.0055 | matching |
| quality innovation parameter | $\xi$ | 0.69 | matching |

Using the original (normalized) data means that we need to solve the non-stationary equilibrium of a growing economy. For this purpose, additional assumptions are imposed on the model and data. First, for simplicity, the agent is assumed to have correct expectations about future productivity evolution in both sectors. In the model, output growth comes exclusively from productivity changes. Thus, for the sample horizon 1970-2006, projections of $\{A_t, B_t\}$ are based on the time series $\{Y_{at}, Y_{bt}\}$. The model is effectively deterministic because the distribution of future states is degenerate. Second, we need to specify the initial and terminal conditions. The initial state composed of total capital stock and quality indices is assumed to take on steady state values if the 1970's sectoral productivity levels stay forever. For the terminal conditions, we assume that the economy goes on for 10 more years after 2006. It is ideal to have a very long model horizon to guarantee that the terminal conditions do not affect behaviors in the sample horizon. However, this truncation of future is necessary for feasible computations and reliable projections of productivity evolutions. Technically, if the out-of-sample horizon is too long, the two projected sectoral productivity time paths diverge so much that numerical computations of the equilibrium are not reliable. In fact, to compute the equilibrium, we implement the Newton's method in C++ with the initial guess at steady state values of each year (Appendix A.5).

**Uncritical Parameters.** From Proposition 3.3, we know that some parameters are not important to the relative performance between the sectors. Those uncritical in the (relative) quality innovation process are $\{\lambda,\sigma,\delta,\nu,\phi, \mu,\gamma\}$. First, the time discount factor $\lambda$ and capital depreciation rate $\delta$ are rounded-off values based on the US calibration exercise by Cooley & Prescott (1995). Second, the CRRA curvature parameter $\sigma$ is often chosen to be unit so that the utility function is logarithmic. We choose another value for this parameter to avoid any special effects possibly linked to the logarithmic form. Third, specification of the contribution factor is based on a derivation of the FOC that $\nu = p_{at}x_{at}/(p_{at}x_{at} + p_{bt}x_{bt})$. In words, $\nu$ is the share of sector 1 in investment market value. This parameter should not be too far from the budget share for goods in consumption, which is 0.47 on average in 1970-2006 (NIPA 1.5.5.). This numerical value is not compatible with the computation of the equilibrium. Via experiments, we fix $\nu$ at 0.42, i.e. goods contribute a smaller share than services in capital accumulation. This at first seems contradictory to the common sense that services cannot be accumulated. However, it should be borne in mind that almost all economic activities have some services contents like electricity, transportation, insurance, and finance. In general, these various services play a crucial role in production, location, purchases, and consumption. Hence services are embodied in the capital stock. Fourth, again, we assume the investment scale factor to be unit for simplicity. Fifth, in principle, the parameters $\mu$ and $\gamma$ can be specified based on income shares (NIPA 6.1). In the data, capital shares in goods and services industries are respectively 0.28 and 0.35 in 1970-2000, which means sector 2 (services) employs relatively more labor than sector 1 (goods). However, the difference between those two values is large enough to spoil equilibrium computations. The economy-wide capital share is 0.33. Based on this, we choose $\mu = 0.32$ and $\gamma = 0.34$.

**Critical Parameters.** Seeking $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$ is an interesting exercise because they are either new in the literature or mysterious in this application. As mentioned earlier, we rely on moment matching to pin them down, where the objective function is the sum of squared differences between data and model moments. Unfortunately, as the data set is limited, we do not have enough moments to identify all of these parameters at the same time. Experiments on the discretized parameter space show that $\eta_\alpha$ and $\eta_\beta$ should be small but cannot be zero, and $\eta_\beta > \eta_\alpha$. We assume that $\eta_\beta = 0.0055$ and $\eta_\alpha = 0.0045$, i.e. quality of services is easier to depreciate than quality of goods.

Here are some details about the moments. We specify the time paths of $A_t$ and $B_t$ as smooth trends based on those of $Y_{at}$ and $Y_{bt}$. In addition, growth in $B_t$ is then scaled down by a factor of 0.98 to make the model budget share for services close to the data counterpart in equilibrium. Effectively, we fix the time series of $B_t/A_t$. The moments reflect responses of endogenous relative objects to the evolution of $B_t/A_t$. The first moment is the angle between linear trends of $Y_{bt}/Y_{at}$ and $S_{bt}$. The second moment is the slope of the linear trend in $S_{bt}$. Changes in $\theta$ and $\xi$ do vary these moments. There is a subtle and negligible difference between data and model moments: data moments are based on linear projections of the original data, whereas model moments are conditional on linear projections of trends in the original data (output average growth rates).

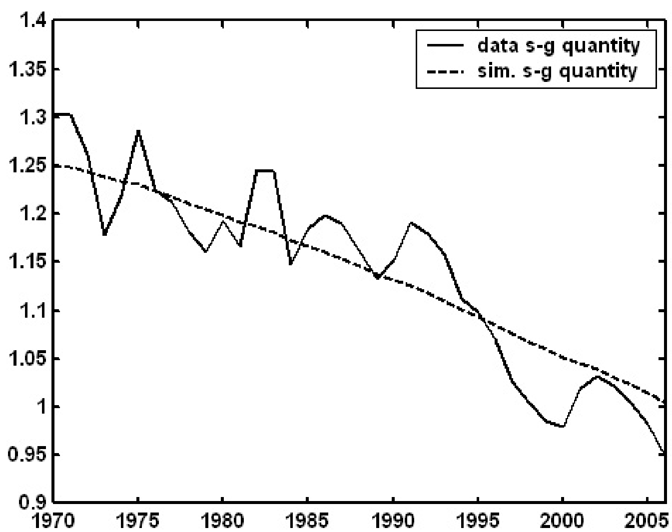
Examinations of the objective function reveal that we do not have a globally convex programming problem. This prevents us from using the simulated method of moments to estimate these two parameters. To overcome this challenge, the exercise has two steps. First is finding a good guess, and second is iteratively searching for $\theta$ and $\xi$. We discretize the parameter space to have guesses of $\theta$ and $\xi$. In fact, the final parameters are not far from the guesses.

Specifically $\theta = -7.0$, $\xi = 0.69$, and note that $\theta < 1/(1+\xi)$. The corresponding elasticity of substitution between goods and services is $-0.125$. This means goods and services are hard to be substituted.

### 4.3 Simulated Time Paths

To pin down parameters, we rely on data trends and specify smooth time paths of $A_t$ and $B_t$. Here, we perturb the time series of $A_t$ and $B_t$ around the smooth trends to mimic the evolution of actual $Y_{at}$ and $Y_{bt}$; then generate some time paths of interest; and compare those with data. In overall, simulated objects match the data trends quite well (Figures 4-7). However, there are some features in the data that the model does not mimic satisfactorily.

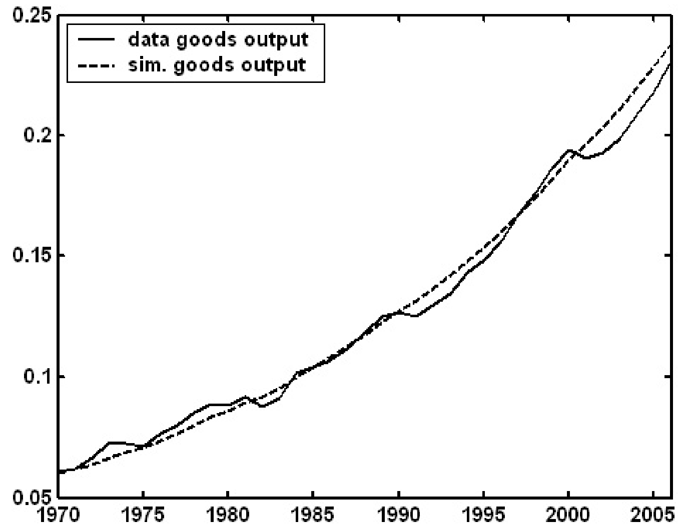
Figure 4: Services-goods relative quantity, 1970-2006




*Note*: simulated quantity is renormalized for trend comparisons.

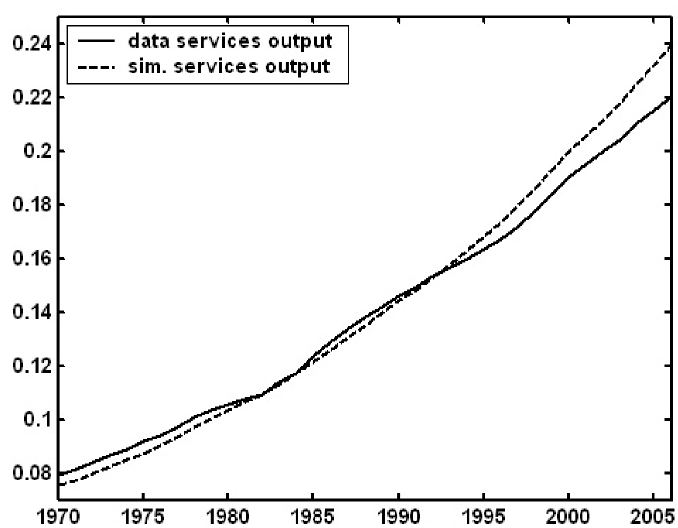
Here are some detailed observations. First, our projections of the productivity time paths can generate the extent of economic growth in 1970-2006 data (Figures 5 and 6). In addition, the simulated evolution patterns in output can mimic the reality. However, as the perturbations around productivity trends are small, some simulated time paths are smoother than the data evolutions like those in Figures 4 and 5. Second, the simulated services-goods relative price matches the data in both trend and pattern to a large extent (Figure 7). This again supports the productivity projections based on actual output time series. Third, the trend in simulated budget share for services does match the reality (Figure 8). However, even though the simulated and data time series have large positive correlation (0.98), their patterns are different. Specifically,

Figure 5: Data and simulated goods quantity, 1970-2006



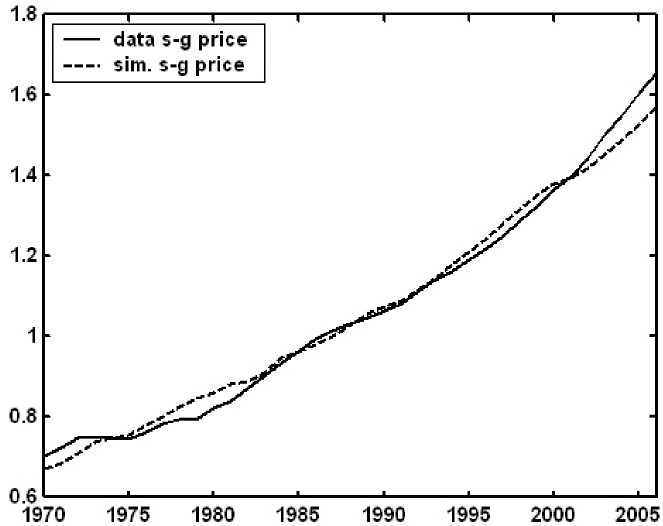

*Note*: data output is renormalized for trend comparisons.

Figure 6: Data and simulated services quantity, 1970-2006



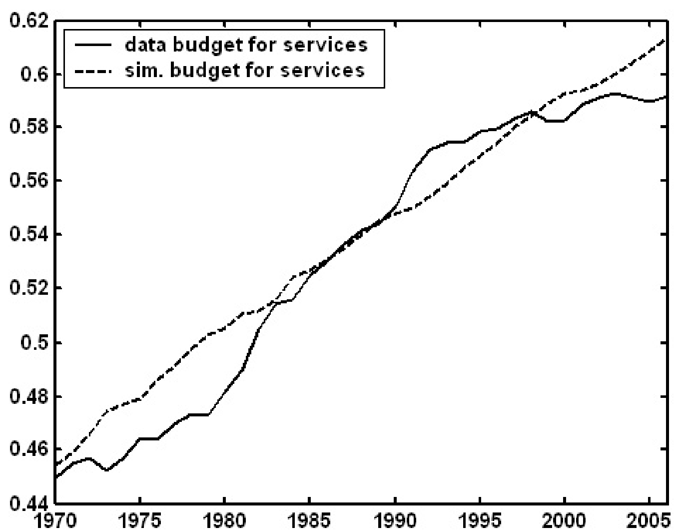

*Note*: data output is renormalized for trend comparisons.

Figure 7: Services-goods relative price, 1970-2006




*Note*: simulated price is renormalized for trend comparisons.

Figure 8: Budget share for services, 1970-2006



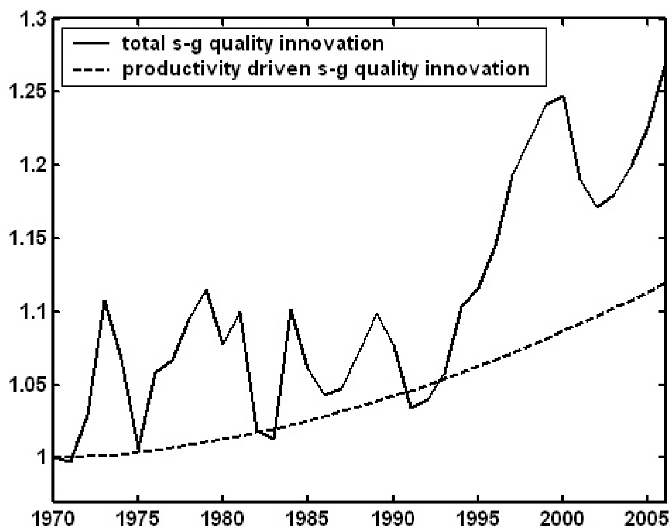
while the data time series has the sigmoid shape, the simulated one shows a clear positive trend. In addition, the latter is less volatile than the former.

In overall, simulated variables have the same correlation patterns presented in Table 2 (for little substitutability). An increase in productivity $B/A$ will induces a transfer of labor and capital to sector 1. Thus, relative productivity and relative quality are negatively correlated.

### 4.4 Productivity driven vs. Total Quality Innovation

The previous comparisons show that the model can mimic the extent of growth in observable objects. This implies we can generate a reliable trend of services relative quality driven by productivity changes (Figure 9).

Figure 9: Productivity driven vs. total quality innovation, 1970-2006




*Note*: both are services quality relative to goods quality; total quality innovation is based on Nguyen (2007b); $\theta = -7.0$; 1970's values are normalized to 1; we compare the trends of two time series. It suggests that productivity driven quality can play a big role in total quality. In addition, total quality innovation should not be interpreted purely as a taste shock. As total quality has other driving forces, e.g. product competition, besides productivity, we do not expect productivity driven quality to fully match the trend of total quality.

To answer the second question about the quantitative importance of productivity driven quality, we compare, by means of Figure 9, the growth trend of this quality concept with that of total quality which is already presented in Figure 1. Recall that total quality innovation is based on the quality inference method in Nguyen (2007b), which can be interpreted as an accounting exercise. Specifically, quantity changes alone cannot fully explain evolution of the relative price of services, and the remaining part is caused by total quality innovation. Thus, total quality innovation varies due to all possible sources like randomly developed ideas for quality improvement as well as productivity variations.

In Figure 9, both relative quality concepts move upwards over time, i.e. quality of services increases in relation to that of goods. However, total quality grows faster than productivity driven quality. When comparing the trends, from 1970-2006, productivity driven quality accounts for about two fifths to one half of total quality growth. Thus, productivity evolution and its quality responses do play a significant role in general quality development. This also means we can see how important other factors, which are not tied to productivity as in the baseline model, collectively are. At the aggregate level, those other factors, e.g. random innovation ideas and quality races, can be treated as exogenous and introduced into the laws of motion (7) and (8). We do not attempt this extension here. In support of the results in Nguyen (2007b), Figure 9 suggests that total quality innovation should not be interpreted purely as a taste shock.

# 5 Conclusion

In this study, we examine how relative productivity affects relative quality in the context of two-sector competitive growth and RBC models, and how much variation in relative quality of the services sector in the US can be accounted for with productivity changes alone. Differing from the variety-growth and quality-ladder literature, this study does not use market power to explain why quality varies. Specifically, labor is used for both quantity production and quality innovation. As quantity and quality can be substitutable, a change in productivity induces a reallocation of labor, leading to quality variation.

Our study finds that productivity's effect on quality depends on two key parameters, which govern how substitutable the products are and how easy it is to improve quality. Specifically, productivity and quality have a negative correlation for low-range substitutability and a positive correlation for medium-range substitutability, where the upper bound of the medium range negatively depends on the innovation parameter. The model is then applied to the goods and services sectors of the US from 1970-2006 using aggregate data. The main empirical result suggests that productivity driven quality can play a significant role in total quality. In addition, the parameter estimates imply a negative correlation between productivity and quality.

The quantitative importance of productivity driven quality in general quality development calls for attention at this type of quality response in growth and RBC models. For example, we should base our optimal tax policies on a model of the economy where quality responds to productivity rather than a model where quality responses are omitted. At least, this type of quality response can be included in sensitivity analyses or thought experiments to see its missing is benign or not.

# References


Aghion, P. & Howitt, P. (1992), 'A model of growth through creative destruction', *Econometrica* **60**(2), 323–351.

Christiano, L. J. (2002), 'Solving dynamic equilibrium models by a method of undetermined coefficients', *Computational Economics* **20**(1), 21–55.

Cooley, T. & Prescott, E. (1995), 'Economic growth and business cycles', *Frontiers of Business Cycle Research* . Princeton: Princeton University Press.

Grossman, G. M. & Helpman, E. (1991), 'Quality ladders in the theory of growth', *Review of Economic Studies* **58**(1), 43–61.

Kehoe, T. J. & Ruhl, K. J. (2005), 'Sudden stops, sectoral reallocation and the real exchange rates', Manuscript, University of Minnesota & University of Texas at Austin.

Kydland, F. & Prescott, E. (1982), 'Time to build and aggregate fluctuations', *Econometrica* **50**(6), 1345–70.

Nguyen, T. Q. (2007a), 'Technological progress in races for product supremacy', Manuscript, University of Texas at Austin.

Nguyen, T. Q. (2007b), 'Separating productivity and quality effects from relative prices', Manuscript, University of Texas at Austin.

# A Appendix

## A.1 Social Planner Problem: Necessary Conditions

We are solving the dynamic programming problem laid out in Definition 2.1. There are some notes about the choice variables in Table A.1. It is noted that the effective number of decisions is eight.

Table A.1: Notes about choice variables

| description | range | notes |
|---|---|---|
| labor for $Y_{at}$ | $0 < l_{at} < 1$ | $l_{at} > 0$ for $\mu \in (0,1)$ |
| labor for $Y_{bt}$ | $0 < l_{bt} < 1$ | $l_{bt} > 0$ for $\gamma \in (0,1)$ |
| labor for $\alpha_t$ | $0 < n_{at} < 1$ | $n_{at} > 0$ for $\xi \in (0,1)$ |
| labor for $\beta_t$ | $0 < n_{bt} < 1$ | $n_{bt} > 0$ for $\xi \in (0,1)$ |
| capital for $Y_{at}$ | $0 < k_{at} < k_t$ | $k_{at} > 0$ for $\mu \in (0,1)$ |
| capital for $Y_{bt}$ | $0 < k_{bt} < k_t$ | $k_{bt} > 0$ for $\gamma \in (0,1)$ |
| consumption of $a$ | $0 < a_t < Y_{at}$ | replaced, $a_t = Y_{at} - x_{at}$ |
| consumption of $b$ | $0 < b_t < Y_{bt}$ | replaced, $b_t = Y_{bt} - x_{bt}$ |
| investment from $a$ | $0 < x_{at} < Y_{at}$ | $x_{at} > 0$ for $\nu \in (0,1)$ |
| investment from $b$ | $0 < x_{bt} < Y_{bt}$ | $x_{bt} > 0$ for $\nu \in (0,1)$ |

Recall that $\omega_t = \{\alpha_{t-1}, \beta_{t-1}, k_t, A_t, B_t\}$. Let $\{\mu_{1t}, \mu_{2t}\}$ be the Lagrangian multipliers for the capital and labor constraints in period $t$. The Bellman equation in (25) can be rewritten as

$$\begin{aligned} V(\omega_t) &= \max_{C_{1t}, C_{2t}} L(\omega_t), \qquad \text{(A.1)} \\ L(\omega_t) &= u(Y_{at} - x_{at}, Y_{bt} - x_{bt}; \alpha_t, \beta_t) + \lambda E_t V(\omega_{t+1}) \\ &\quad + \mu_{1t}(1 - l_{at} - l_{bt} - n_{at} - n_{bt}) \\ &\quad + \mu_{2t}(k_t - k_{at} - k_{bt}). \end{aligned}$$

To simplify the upcoming expressions, let

$$c_t = \left[\alpha_t^\theta (Y_{at} - x_{at})^\theta + \beta_t^\theta (Y_{bt} - x_{bt})^\theta\right]^{1/\theta} \qquad \text{(A.2)}$$

$$d_t = \left[\alpha_t^\theta (Y_{at} - x_{at})^\theta + \beta_t^\theta (Y_{bt} - x_{bt})^\theta\right]^{1/\theta - 1} \qquad \text{(A.3)}$$

$$u_t = \frac{c_t^{1-\sigma} - 1}{1 - \sigma}. \qquad \text{(A.4)}$$

The necessary and also sufficient conditions are

$l_{at}$ :

$$c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta - 1} (1 - \mu) Y_{at}/l_{at} = \mu_{1t} \qquad \text{(A.5)}$$

$l_{bt}$ :

$$c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta - 1} (1 - \gamma) Y_{bt}/l_{bt} = \mu_{1t} \qquad \text{(A.6)}$$

$k_{at}$ :

$$c_t^{-\sigma} d_t \alpha_t^{\theta} a_t^{\theta-1} \mu Y_{at}/k_{at} = \mu_{2t} \tag{A.7}$$

$k_{bt}$ :

$$c_t^{-\sigma} d_t \beta_t^{\theta} b_t^{\theta-1} \gamma Y_{bt}/k_{bt} = \mu_{2t} \tag{A.8}$$

$n_{at}$ :

$$\begin{gathered} c_t^{-\sigma} d_t a_t^{\theta} \alpha_t^{\theta-1} \xi \left(n_{at}\right)^{\xi-1} \\ +\lambda E_t \left[c_{t+1}^{-\sigma} d_{t+1} a_{t+1}^{\theta} \alpha_{t+1}^{\theta-1} \left(1-\eta_{\alpha}\right) \xi \left(n_{at}\right)^{\xi-1}\right] = \mu_{1t} \end{gathered} \tag{A.9}$$

$n_{bt}$ :

$$\begin{gathered} c_t^{-\sigma} d_t b_t^{\theta} \beta_t^{\theta-1} \xi \left(n_{bt}\right)^{\xi-1} \\ +\lambda E_t \left[c_{t+1}^{-\sigma} d_{t+1} b_{t+1}^{\theta} \beta_{t+1}^{\theta-1} \left(1-\eta_{\beta}\right) \xi \left(n_{bt}\right)^{\xi-1}\right] = \mu_{1t} \end{gathered} \tag{A.10}$$

$x_{at}$ :

$$\begin{gathered} c_t^{-\sigma} d_t \alpha_t^{\theta} a_t^{\theta-1} = \\ +\lambda E_t \left[c_{t+1}^{-\sigma} d_{t+1} \alpha_{t+1}^{\theta} a_{t+1}^{\theta-1} \mu \left(Y_{at+1}/k_{at+1}\right) \nu \left(\phi x_{at}^{\nu-1} x_{bt}^{1-\nu}\right)\right] \end{gathered} \tag{A.11}$$

$x_{bt}$ :

$$\begin{gathered} c_t^{-\sigma} d_t \beta_t^{\theta} b_t^{\theta-1} = \\ +\lambda E_t \left[c_{t+1}^{-\sigma} d_{t+1} \beta_{t+1}^{\theta} b_{t+1}^{\theta-1} \gamma \left(Y_{bt+1}/k_{bt+1}\right) \left(1-\nu\right) \left(\phi x_{at}^{\nu} x_{bt}^{-\nu}\right)\right] \end{gathered} \tag{A.12}$$

$\mu_{1t}$ :

$$l_{at} + l_{bt} + n_{at} + n_{bt} = 1;\ \mu_{1t} > 0 \tag{A.13}$$

$\mu_{2t}$ :

$$k_{at} + k_{bt} = k_t;\ \mu_{2t} > 0. \tag{A.14}$$

## A.2 Social Planner Problem: the Steady State

We are now solving for the nonstochastic steady state which satisfies

$$\left\{ \begin{array}{ll} \text{state:} & A_t = A, B_t = B, \alpha_t = \alpha, \beta_t = \beta, k_t = k; \\ \text{labor:} & l_{at} = l_a, l_{bt} = l_b, n_{at} = n_a, n_{bt} = n_b; \\ \text{capital:} & k_{at} = k_a, k_{bt} = k_b; \\ \text{output:} & Y_{at} = Y_a, Y_{bt} = Y_b; \\ \text{uses:} & a_t = a, b_t = b, x_{at} = x_a, x_{bt} = x_b; \\ \text{multipliers:} & \mu_{1t} = \mu_1, \mu_{2t} = \mu_2. \end{array} \right.$$

1) There are 13 conditions for 13 unknowns $(l_a, l_b)$, $(n_a, n_b)$,$(k, k_a, k_b)$, $(\alpha, \beta)$, $(x_a, x_b)$, $(\mu_1, \mu_2)$

$$c^{-\sigma} d \alpha^{\theta} a^{\theta-1} \left(1-\mu\right) \left(Y_a/l_a\right) = \mu_1 \tag{A.15}$$

$$c^{-\sigma} d \beta^{\theta} b^{\theta-1} \left(1-\gamma\right) \left(Y_b/l_b\right) = \mu_1 \tag{A.16}$$

$$c^{-\sigma} d \alpha^{\theta} a^{\theta-1} \mu \left(Y_a/k_a\right) = \mu_2 \tag{A.17}$$

$$c^{-\sigma} d \beta^{\theta} b^{\theta-1} \gamma \left(Y_b/k_b\right) = \mu_2 \tag{A.18}$$

$$c^{-\sigma} d a^{\theta} \alpha^{\theta-1} \left[1 + \lambda \left(1-\eta_{\alpha}\right)\right] \xi \left(n_a\right)^{\xi-1} = \mu_1 \tag{A.19}$$

$$c^{-\sigma} d b^{\theta} \beta^{\theta-1} \left[1+\lambda\left(1-\eta_{\beta}\right)\right] \xi \left(n_{b}\right)^{\xi-1}=\mu_{1} \tag{A.20}$$

$$1=\lambda \mu\left(Y_{a} / k_{a}\right) \nu\left(\phi x_{a}^{\nu-1} x_{b}^{1-\nu}\right) \tag{A.21}$$

$$1=\lambda \gamma\left(Y_{b} / k_{b}\right)(1-\nu)\left(\phi x_{a}^{\nu} x_{b}^{-\nu}\right) \tag{A.22}$$

$$n_{a}=\left(\eta_{\alpha} \alpha\right)^{1 / \xi} \tag{A.23}$$

$$n_{b}=\left(\eta_{\beta} \beta\right)^{1 / \xi} \tag{A.24}$$

$$\phi x_{a}^{\nu} x_{b}^{1-\nu}=\delta k \tag{A.25}$$

$$l_{a}+l_{b}=1-\left[\left(\eta_{\alpha} \alpha\right)^{1 / \xi}+\left(\eta_{\beta} \beta\right)^{1 / \xi}\right] \tag{A.26}$$

$$k_{a}+k_{b}=k . \tag{A.27}$$

2) Substitute $n_a$ into (A.19) and $n_b$ into (A.20). In addition, the multipliers $\mu_1$ and $\mu_2$ can be eliminated. The system is collapsed into a new one with 9 equations in 9 unknowns $(l_a, l_b)$, $(k, k_a, k_b)$, $(\alpha, \beta)$, $(x_a, x_b)$

$$\alpha^{1 / \xi}(1-\mu)\left(Y_{a} / l_{a}\right)=a\left[1+\lambda\left(1-\eta_{\alpha}\right)\right] \xi \eta_{\alpha}^{(1-1 / \xi)} \tag{A.28}$$

$$\beta^{1 / \xi}(1-\gamma)\left(Y_{b} / l_{b}\right)=b\left[1+\lambda\left(1-\eta_{\beta}\right)\right] \xi \eta_{\beta}^{(1-1 / \xi)} \tag{A.29}$$

$$\alpha^{\theta} a^{\theta-1}(1-\mu)\left(Y_{a} / l_{a}\right)=\beta^{\theta} b^{\theta-1}(1-\gamma)\left(Y_{b} / l_{b}\right) \tag{A.30}$$

$$\alpha^{\theta} a^{\theta-1} \mu\left(Y_{a} / k_{a}\right)=\beta^{\theta} b^{\theta-1} \gamma\left(Y_{b} / k_{b}\right) \tag{A.31}$$

$$1=\lambda \mu\left(Y_{a} / k_{a}\right) \nu\left(\phi x_{a}^{\nu-1} x_{b}^{1-\nu}\right) \tag{A.32}$$

$$1=\lambda \gamma\left(Y_{b} / k_{b}\right)(1-\nu)\left(\phi x_{a}^{\nu} x_{b}^{-\nu}\right) \tag{A.33}$$

$$\phi x_{a}^{\nu} x_{b}^{1-\nu}=\delta k \tag{A.34}$$

$$l_{a}+l_{b}=1-\left[\left(\eta_{\alpha} \alpha\right)^{1 / \xi}+\left(\eta_{\beta} \beta\right)^{1 / \xi}\right] \tag{A.35}$$

$$k_{a}+k_{b}=k . \tag{A.36}$$

3) We transform some variables to simplify the system. Let

$$S=k_{b} / k \tag{A.37}$$

$$Q=k_{b} / l_{b} \tag{A.38}$$

$$X_{a}=x_{a} / Y_{a} \tag{A.39}$$

$$X_{b}=x_{b} / Y_{b} . \tag{A.40}$$

From (A.30) and (A.31)

$$\frac{k_{b}}{k_{a}}=\left[\frac{\gamma(1-\mu)}{\mu(1-\gamma)}\right] \frac{l_{b}}{l_{a}} . \tag{A.41}$$

From (A.38) and (A.41)

$$\frac{k_a}{l_a} = \left[\frac{\mu(1-\gamma)}{\gamma(1-\mu)}\right] Q. \tag{A.42}$$

From (A.42)

$$\frac{Y_a}{l_a} = A\left[\frac{\mu(1-\gamma)}{\gamma(1-\mu)}\right]^{\mu} Q^{\mu}, \tag{A.43}$$

$$\frac{Y_a}{k_a} = A\left[\frac{\mu(1-\gamma)}{\gamma(1-\mu)}\right]^{\mu-1} Q^{\mu-1}. \tag{A.44}$$

From (A.38)

$$\frac{Y_b}{l_b} = BQ^{\gamma}, \tag{A.45}$$

$$\frac{Y_b}{k_b} = BQ^{\gamma-1}. \tag{A.46}$$

From (A.37) and (A.41), we have

$$\frac{S}{1-S} = \left[\frac{\gamma(1-\mu)}{\mu(1-\gamma)}\right] \frac{l_b}{l_a}$$

$$\frac{l_a}{\gamma(1-\mu)(1-S)} = \frac{l_b}{\mu(1-\gamma)S} = \frac{1-\left[(\eta_\alpha\alpha)^{1/\xi} + (\eta_\beta\beta)^{1/\xi}\right]}{\gamma(1-\mu)+(\mu-\gamma)S}$$

$$l_a = \frac{\gamma(1-\mu)(1-S)}{\gamma(1-\mu)+(\mu-\gamma)S}\left\{1-\left[(\eta_\alpha\alpha)^{1/\xi} + (\eta_\beta\beta)^{1/\xi}\right]\right\}, \tag{A.47}$$

$$l_b = \frac{\mu(1-\gamma)S}{\gamma(1-\mu)+(\mu-\gamma)S}\left\{1-\left[(\eta_\alpha\alpha)^{1/\xi} + (\eta_\beta\beta)^{1/\xi}\right]\right\}. \tag{A.48}$$

From (A.28) and (A.29)

$$a = \frac{\alpha^{1/\xi}(1-\mu)(Y_a/l_a)}{[1+\lambda(1-\eta_\alpha)]\xi\eta_\alpha^{(1-1/\xi)}}, \tag{A.49}$$

$$b = \frac{\beta^{1/\xi}(1-\gamma)(Y_b/l_b)}{[1+\lambda(1-\eta_\beta)]\xi\eta_\beta^{(1-1/\xi)}}. \tag{A.50}$$

Based on (A.43)-(A.46) and (A.49)-(A.50), equation (A.31) can be rewritten as

$$\frac{\alpha^\theta\mu(Y_a/k_a)}{\beta^\theta\gamma(Y_b/k_b)} = \frac{b^{\theta-1}}{a^{\theta-1}}$$

$$\frac{\alpha^\theta\mu}{\beta^\theta\gamma}\frac{(Y_a/k_a)}{(Y_b/k_b)} = \left[\frac{(1-\mu)[1+\lambda(1-\eta_\beta)]\eta_\beta^{(1-1/\xi)}}{(1-\gamma)[1+\lambda(1-\eta_\alpha)]\eta_\alpha^{(1-1/\xi)}}\frac{\alpha^{1/\xi}}{\beta^{1/\xi}}\frac{(Y_a/l_a)}{(Y_b/l_b)}\right]^{1-\theta}$$

$$\frac{(Y_a/k_a)}{(Y_b/k_b)}\frac{(Y_a/l_a)^{\theta-1}}{(Y_b/l_b)^{\theta-1}}\frac{\alpha^\theta}{\beta^\theta}=\frac{\gamma}{\mu}\left[\frac{(1-\mu)\left[1+\lambda\left(1-\eta_\beta\right)\right]\eta_\beta^{(1-1/\xi)}}{(1-\gamma)\left[1+\lambda\left(1-\eta_\alpha\right)\right]\eta_\alpha^{(1-1/\xi)}}\frac{\alpha^{1/\xi}}{\beta^{1/\xi}}\right]^{1-\theta}$$

$$Q^{\theta(\mu-\gamma)}\left(\frac{\alpha}{\beta}\right)^{\theta-(1-\theta)/\xi}=C_1, \tag{A.51}$$

where

$$C_1=\frac{B^\theta}{A^\theta}\left[\frac{\left[1+\lambda\left(1-\eta_\beta\right)\right]\eta_\beta^{(1-1/\xi)}}{\left[1+\lambda\left(1-\eta_\alpha\right)\right]\eta_\alpha^{(1-1/\xi)}}\right]^{1-\theta}\frac{\gamma^{\theta\mu}(1-\mu)^{\theta(\mu-1)}}{\mu^{\theta\mu}(1-\gamma)^{\theta(\mu-1)}}. \tag{A.52}$$

Equation (A.32) can be rewritten as

$$1=\frac{\lambda\mu\left(Y_a/k_a\right)\nu\left(\phi x_a^\nu x_b^{1-\nu}\right)}{x_a}$$

$$\frac{\left(Y_a/k_a\right)\nu\left(\phi x_a^\nu x_b^{1-\nu}\right)}{X_aY_a}=\frac{1}{\lambda\mu}$$

$$\frac{\nu\delta k}{X_ak_a}=\frac{1}{\lambda\mu}$$

$$X_a=\frac{\nu\delta\lambda\mu}{1-S}. \tag{A.53}$$

Equation (A.33) can be rewritten as

$$1=\frac{\lambda\gamma\left(Y_b/k_b\right)(1-\nu)\left(\phi x_a^\nu x_b^{1-\nu}\right)}{x_b}$$

$$\frac{\left(Y_b/k_b\right)(1-\nu)\left(\phi x_a^\nu x_b^{1-\nu}\right)}{X_bY_b}=\frac{1}{\lambda\gamma}$$

$$\frac{(1-\nu)\,\delta k}{X_bk_b}=\frac{1}{\lambda\gamma}$$

$$X_b=\frac{(1-\nu)\,\delta\lambda\gamma}{S}. \tag{A.54}$$

4) We now solve for $Q$. Equation (A.34) can be rewritten as

$$\phi\left(X_aY_a\right)^\nu\left(X_bY_b\right)^{1-\nu}=\delta k \tag{A.55}$$

$$\phi\left[\frac{\nu\delta\lambda\mu}{1-S}\right]^\nu\left[\frac{(1-\nu)\,\delta\lambda\gamma}{S}\right]^{1-\nu}\left(\frac{Y_a}{k_a}k_a\right)^\nu\left(\frac{Y_b}{k_b}k_b\right)^{1-\nu}=\delta k$$

$$\lambda\phi A^\nu B^{1-\nu}\nu^\nu(1-\nu)^{1-\nu}\mu^{\mu\nu}\gamma^{1-\mu\nu}\left[\frac{1-\gamma}{1-\mu}\right]^{(\mu-1)\nu}Q^\Lambda=1.$$

Thus

$$Q=C_2, \tag{A.56}$$

where

$$C_2 = \left[ \lambda \phi A^{\nu} B^{1-\nu} \nu^{\nu} (1-\nu)^{1-\nu} \mu^{\mu\nu} \gamma^{1-\mu\nu} \left[ \frac{1-\gamma}{1-\mu} \right]^{(\mu-1)\nu} \right]^{-1/\Lambda}, \tag{A.57}$$

$$\Lambda = \nu (\mu - 1) + (1-\nu)(\gamma - 1) \neq 0.$$

5) We find the expression for $\beta/\alpha$. From (A.51) and (A.56)

$$\left( \frac{\beta}{\alpha} \right)^{\theta - (1-\theta)/\xi} = \frac{C_2^{\theta(\mu-\gamma)}}{C_1}$$

$$\frac{\beta}{\alpha} = C_3, \tag{A.58}$$

where

$$C_3 = \left[ \frac{C_2^{\theta(\mu-\gamma)}}{C_1} \right]^{1/[\theta-(1-\theta)/\xi]}, \ \theta (1+\xi) \neq 1. \tag{A.59}$$

6) We will find $(S, X_a, X_b)$. Equations (A.28) and (A.29) are rewritten as

$$\frac{\alpha^{1/\xi} (1-\mu)}{[1+\lambda (1-\eta_\alpha)] \xi \eta_\alpha^{(1-1/\xi)}} = (1 - X_a) l_a, \tag{A.60}$$

$$\frac{\beta^{1/\xi} (1-\gamma)}{[1+\lambda (1-\eta_\beta)] \xi \eta_\beta^{(1-1/\xi)}} = (1 - X_b) l_b. \tag{A.61}$$

Based on (A.47)-(A.48) and (A.53)-(A.54), we divide (A.61) by (A.60)

$$\frac{[1+\lambda (1-\eta_\alpha)] \eta_\alpha^{(1-1/\xi)} (1-\gamma)}{[1+\lambda (1-\eta_\beta)] \eta_\beta^{(1-1/\xi)} (1-\mu)} \left( \frac{\beta}{\alpha} \right)^{1/\xi} = \frac{\left[ 1 - \frac{(1-\nu)\delta\lambda\gamma}{S} \right]}{\left[ 1 - \frac{\nu\delta\lambda\mu}{1-S} \right]} \frac{\mu (1-\gamma) S}{\gamma (1-\mu)(1-S)}$$

$$\frac{\gamma [1+\lambda (1-\eta_\alpha)] \eta_\alpha^{(1-1/\xi)}}{\mu [1+\lambda (1-\eta_\beta)] \eta_\beta^{(1-1/\xi)}} (C_3)^{1/\xi} = \frac{[S - (1-\nu) \delta\lambda\gamma]}{[1 - S - \nu\delta\lambda\mu]}.$$

Let

$$C_4 = \frac{\gamma [1+\lambda (1-\eta_\alpha)] \eta_\alpha^{(1-1/\xi)}}{\mu [1+\lambda (1-\eta_\beta)] \eta_\beta^{(1-1/\xi)}} (C_3)^{1/\xi}, \tag{A.62}$$

and the expression for $S$ follows

$$\frac{S - (1-\nu)\delta\lambda\gamma}{1 - S - \nu\delta\lambda\mu} = C_4$$

$$S = \frac{C_4 (1 - \nu\delta\lambda\mu) + (1-\nu)\delta\lambda\gamma}{1 + C_4}. \tag{A.63}$$

Plug $S$ into (A.53) to find $X_a$ and into (A.54) to find $X_b$.

7) At this point, all of the transformed variables are known, i.e. $S$, $Q$, $X_a$, $X_b$ can be defined as explicit functions of the parameters. Other variables can now be derived. From (A.47) and (A.60), we have

$$\alpha^{1/\xi} = C_5 \left\{1 - \left[(\eta_\alpha \alpha)^{1/\xi} + (\eta_\beta \beta)^{1/\xi}\right]\right\},$$

where

$$C_5 = \frac{\left[1 + \lambda (1 - \eta_\alpha)\right] \xi \eta_\alpha^{(1-1/\xi)} (1 - X_a) \gamma (1 - S)}{\gamma (1 - \mu) + (\mu - \gamma) S}; \tag{A.64}$$

and

$$\alpha^{1/\xi} = C_5 - C_5 (\eta_\alpha)^{1/\xi} \alpha^{1/\xi} - C_5 (\eta_\beta C_3)^{1/\xi} \alpha^{1/\xi}$$

$$\left[1 + C_5 (\eta_\alpha)^{1/\xi} + C_5 (\eta_\beta C_3)^{1/\xi}\right] \alpha^{1/\xi} = C_5.$$

Thus, the steady-state quality indices are

$$\alpha = \left[\frac{C_5}{1 + C_5 (\eta_\alpha)^{1/\xi} + C_5 (\eta_\beta C_3)^{1/\xi}}\right]^\xi, \tag{A.65}$$

$$\beta = C_3 \left[\frac{C_5}{1 + C_5 (\eta_\alpha)^{1/\xi} + C_5 (\eta_\beta C_3)^{1/\xi}}\right]^\xi. \tag{A.66}$$

Based on $\alpha$ and $\beta$, we can back out $n_a$ from (A.23), $n_b$ from (A.24), $l_a$ from (A.47), and $l_b$ from (A.48). Next, we can compute $Y_a$ from (A.43), $Y_b$ from (A.45), $a$ from (A.49), and $b$ from (A.50). Equations (A.39) and (A.40) generate $x_a$ and $x_b$. Based on (A.55), we know $k$. Next are $k_a$ and $k_b$ from (A.37), $\mu_1$ from (A.15) and $\mu_2$ from (A.17).

8) The wage and interest rate are derived here. Normalized $p_a = 1$. The equilibrium wage is based on (A.43) and interest rate on (A.44)

$$w = (1 - \mu) \frac{p_a Y_a}{l_a} = A (1 - \mu) \left[\frac{\mu (1 - \gamma)}{\gamma (1 - \mu)}\right]^{\mu - 1} Q^\mu, \tag{A.67}$$

$$r = \mu \frac{p_a Y_a}{k_a} = A \mu \left[\frac{\mu (1 - \gamma)}{\gamma (1 - \mu)}\right]^{\mu - 1} Q^{\mu - 1}. \tag{A.68}$$

9) There are several objects of special interest: $b/a$, $Y_b/Y_a$, $\beta/\alpha$, $p_{ba}$, $S_b$. We know $\beta/\alpha$ from (A.58). From (A.43), (A.45), (A.49)-(A.50), and (A.58)

$$\frac{b}{a} = \frac{(1 - \gamma) \left[1 + \lambda (1 - \eta_\alpha)\right] \eta_\alpha^{(1-1/\xi)}}{(1 - \mu) \left[1 + \lambda (1 - \eta_\beta)\right] \eta_\beta^{(1-1/\xi)}} \left(\frac{\beta}{\alpha}\right)^{1/\xi} \frac{(Y_b / l_b)}{(Y_a / l_a)}$$

$$\frac{b}{a} = \left[\frac{\gamma^\mu (1 - \gamma)^{1-\mu} \left[1 + \lambda (1 - \eta_\alpha)\right] \eta_\alpha^{(1-1/\xi)}}{\mu^\mu (1 - \mu)^{1-\mu} \left[1 + \lambda (1 - \eta_\beta)\right] \eta_\beta^{(1-1/\xi)}}\right] \frac{B}{A} (C_3)^{1/\xi} C_2^{\gamma - \mu}. \tag{A.69}$$

From (A.37), (A.44) and (A.46)

$$\frac{Y_b}{Y_a} = \left[\frac{\mu(1-\gamma)}{\gamma(1-\mu)}\right]^{1-\mu} \frac{BQ^{\gamma-1}k_b}{AQ^{\mu-1}k_a}$$

$$\frac{Y_b}{Y_a} = \left[\frac{\mu(1-\gamma)}{\gamma(1-\mu)}\right]^{1-\mu} \frac{B}{A} C_2^{\gamma-\mu} \frac{[C_4(1-\nu\delta\lambda\mu)+(1-\nu)\delta\lambda\gamma]}{[1+C_4\nu\delta\lambda\mu-(1-\nu)\delta\lambda\gamma]}. \tag{A.70}$$

From (A.58) and (A.69), the steady-state equilibrium relative price is

$$p_{ba} = \frac{\partial u(a,b;\alpha,\beta)/\partial b}{\partial u(a,b;\alpha,\beta)/\partial a} = \frac{c^{-\sigma} d\beta^\theta b^{\theta-1}}{c^{-\sigma} d\alpha^\theta a^{\theta-1}}$$

$$p_{ba} = \left(\frac{\beta}{\alpha}\right)^\theta \left(\frac{b}{a}\right)^{\theta-1} \tag{A.71}$$

$$p_{ba} = \left[\frac{\gamma^\mu (1-\gamma)^{1-\mu}[1+\lambda(1-\eta_\alpha)]\eta_\alpha^{(1-1/\xi)}}{\mu^\mu (1-\mu)^{1-\mu}[1+\lambda(1-\eta_\beta)]\eta_\beta^{(1-1/\xi)}} \frac{B}{A}\right]^{\theta-1} \frac{C_3^{\theta-(1-\theta)/\xi}}{C_2^{(\gamma-\mu)(1-\theta)}}. \tag{A.72}$$

Based on (A.69) and (A.72)

$$S_b = \frac{p_b b}{p_a a + p_b b} = \frac{p_{ba}(b/a)}{1+p_{ba}(b/a)}$$

$$S_b = \frac{C_6}{1+C_6}, \tag{A.73}$$

$$C_6 = \left[\frac{\gamma^\mu (1-\gamma)^{1-\mu}[1+\lambda(1-\eta_\alpha)]\eta_\alpha^{(1-1/\xi)}}{\mu^\mu (1-\mu)^{1-\mu}[1+\lambda(1-\eta_\beta)]\eta_\beta^{(1-1/\xi)}}\right]^\theta \left(\frac{B}{A}\right)^\theta \frac{C_3^{\theta(1+1/\xi)}}{C_2^{(\mu-\gamma)\theta}}. \tag{A.74}$$

These five objects are functions of $C_1$, $C_2$, and $C_3$, where (conditional on $A$)

$$\begin{aligned} C_1 &= const \times (B/A)^\theta \\ C_2 &= const \times (B/A)^{-(1-\nu)/\Lambda} \\ C_3 &= const \times (B/A)^{(1-\mu)\theta/\{[\theta-(1-\theta)/\xi]\Lambda\}}. \end{aligned}$$

After some manipulations, we observe the following qualitative effects

$$sign[\partial(b/a)/\partial(B/A)] = sign[(1-\theta\xi)(\theta\xi+\theta-1)\Lambda] \tag{A.75}$$

$$sign[\partial(\beta/\alpha)/\partial(B/A)] = sign[\theta(\theta\xi+\theta-1)\Lambda] \tag{A.76}$$

$$sign[\partial p_{ba}/\partial(B/A)] = sign[\Lambda] \tag{A.77}$$

$$sign[\partial S_b/\partial(B/A)] = sign[\theta(\theta\xi+\theta-1)\Lambda]. \tag{A.78}$$

## A.3 Log-linearization and Laws of Motion

To approximate the laws of motion of the endogenous objects, we first log-linearize the system of FOCs (A.5)-(A.14). A transformed variable is interpreted as percentage deviation of the original variable from the corresponding nonstochastic steady state value. In notation, $\widehat{x}_t = \log x_t - \log x$, where $x_t$ is the original variable, $\widehat{x}_t$ is the transformed version, and $x$ is the steady state value. For small deviations, $\widehat{x}_t \approx (x_t - x)/x$, and $f(x_t) = f(x) + f'(x)x\widehat{x}_t$. Evolution of the endogenous state variables $\{\alpha_{t-1}, \beta_{t-1}, k_t\}$ are specified in (6)-(8). The law of motion for the exogenous state variables $\{A_t, B_t\}$ is assumed to follow the VAR structure

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} \mu_a (1-\rho_a) \\ \mu_b (1-\rho_b) \end{bmatrix} + \begin{bmatrix} \rho_a & 0 \\ 0 & \rho_b \end{bmatrix} \begin{bmatrix} A_{t-1} \\ B_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{at} \\ \varepsilon_{bt} \end{bmatrix}, \tag{A.79}$$

where $A_t \sim N(\mu_a, \sigma_a^2)$; $B_t \sim N(\mu_b, \sigma_b^2)$; and $\varepsilon_t = [\varepsilon_{at}\ \varepsilon_{bt}]' \sim N(\mathbf{0}, \Sigma)$ with

$$\mathbf{\Sigma} = \begin{bmatrix} \left(1-\rho_a^2\right)\sigma_a^2 & 0 \\ 0 & \left(1-\rho_b^2\right)\sigma_b^2 \end{bmatrix}. \tag{A.80}$$

We can substitute out the multipliers $\mu_{1t}$ and $\mu_{2t}$. After some simple manipulations, the original FOC becomes

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t a_t^\theta \alpha_t^{\theta-1} \xi (n_{at})^{\xi-1} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} (1-\mu) Y_{at}/l_{at} \\ +\lambda c_{t+1}^{-\sigma} d_{t+1} a_{t+1}^\theta \alpha_{t+1}^{\theta-1} (1-\eta_\alpha) \xi (n_{at})^{\xi-1} \end{array} \right\} = 0 \tag{A.81}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t b_t^\theta \beta_t^{\theta-1} \xi (n_{bt})^{\xi-1} - c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} (1-\gamma) Y_{bt}/l_{bt} \\ +\lambda c_{t+1}^{-\sigma} d_{t+1} b_{t+1}^\theta \beta_{t+1}^{\theta-1} (1-\eta_\beta) \xi (n_{bt})^{\xi-1} \end{array} \right\} = 0 \tag{A.82}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} (1-\mu) Y_{at}/l_{at} \\ -c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} (1-\gamma) Y_{bt}/l_{bt} \end{array} \right\} = 0 \tag{A.83}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} \mu Y_{at}/k_{at} \\ -c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} \gamma Y_{bt}/k_{bt} \end{array} \right\} = 0 \tag{A.84}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} \\ -\lambda c_{t+1}^{-\sigma} d_{t+1} \alpha_{t+1}^\theta a_{t+1}^{\theta-1} \mu (Y_{at+1}/k_{at+1}) \nu \left(\phi x_{at}^{\nu-1} x_{bt}^{1-\nu}\right) \end{array} \right\} = 0 \tag{A.85}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} \\ -\lambda c_{t+1}^{-\sigma} d_{t+1} \beta_{t+1}^\theta b_{t+1}^{\theta-1} \gamma (Y_{bt+1}/k_{bt+1}) (1-\nu) \left(\phi x_{at}^{\nu} x_{bt}^{-\nu}\right) \end{array} \right\} = 0, \tag{A.86}$$

and (4)-(10). In every period $t$, given the state $\{\alpha_{t-1}, \beta_{t-1}, k_t, A_t, B_t\}$, the set of endogenous variables can be collapsed into a minimal set, based on which all the remaining equilibrium objects can be backed out. We choose the minimal set $\{\alpha_t, \beta_t, k_{t+1}, l_{bt}, k_{bt}, x_{bt}\}$. Recall that there are more choice variables. From (4)-(8), the remaining decisions can be defined as follows: $n_{at} = [\alpha_t - (1-\eta_\alpha)\alpha_{t-1}]^{1/\xi}$; $n_{bt} = [\beta_t - (1-\eta_\beta)\beta_{t-1}]^{1/\xi}$; $l_{at} = 1 - l_{bt} - n_{at} - n_{bt}$; $k_{at} = k_t - k_{bt}$;

and $x_{at} = [(k_{t+1} - (1-\delta)k_t)/(\phi x_{bt}^{1-\nu})]^{1/\nu}$. Based on (A.81)-(A.86), we construct a log-linearized system of expectation difference equations

$$E_t \{\zeta_0 z_{t+1} + \zeta_1 z_t + \zeta_2 z_{t-1} + \kappa_0 s_{t+1} + \kappa_1 s_t\} \tag{A.87}$$

where $z_t = [\widehat{\alpha}_t \ \widehat{\beta}_t \ \widehat{k}_{t+1} \ \widehat{l}_{bt} \ \widehat{k}_{bt} \ \widehat{x}_{bt}]'$; $s_t = [\widehat{A}_t \ \widehat{\beta}_t]'$; $\{\zeta_0, \zeta_1, \zeta_2\}$ are $6 \times 6$ matrices; and $\{\kappa_0, \kappa_1\}$ are $6 \times 2$ matrices. In fact, these matrices are the first-order derivatives of the system (A.81)-(A.86) with respect to the corresponding variables in (A.87), and then rescaled by steady state values. Though we can derive these matrices analytically, it is much easier to compute them.

Let $\Gamma(6 \times 6)$ and $\Psi(6 \times 2)$ respectively be the feedback and feedforward matrices. The linear laws of motion of the endogenous objects have the form

$$z_t = \Gamma z_{t-1} + \Psi s_t \tag{A.88}$$

$$s_t = \rho s_{t-1} + \varepsilon_t, \tag{A.89}$$

where matrix $\rho$ is derived from log-linearization of (A.79); $\Gamma$ and $\Psi$ are unknown and solved by the method of undetermined coefficients (Christiano 2002). More specifically, $\Gamma$ and $\Psi$ are respectively solutions of the matrix equations

$$\zeta_0 \Gamma^2 + \zeta_1 \Gamma + \zeta_2 = 0 \tag{A.90}$$

$$\zeta_0 \Gamma \Psi + \zeta_0 \Psi \rho + \zeta_1 \Psi + \kappa_0 \rho + \kappa_1. \tag{A.91}$$

In Section 3, given the specified parameters, these two matrices, which correspond to different values of $\theta$, given $\xi = 2/3$, are

$$\Gamma_{\theta=-3} = \begin{bmatrix} 0.75 & 0.06 & 0.02 & 0 & 0 & 0 \\ 0.06 & 0.71 & 0.03 & 0 & 0 & 0 \\ 0.10 & 0.12 & 0.57 & 0 & 0 & 0 \\ 0.39 & -0.08 & -0.08 & 0 & 0 & 0 \\ 0.27 & -0.26 & 1.00 & 0 & 0 & 0 \\ 1.95 & 2.40 & -7.54 & 0 & 0 & 0 \end{bmatrix}, \ \Psi_{\theta=-3} = \begin{bmatrix} -0.06 & 0.03 \\ 0.03 & -0.07 \\ 0.34 & 0.44 \\ 0.38 & -0.25 \\ 0.37 & -0.35 \\ 6.42 & 9.19 \end{bmatrix};$$

$$\Gamma_{\theta=0.3} = \begin{bmatrix} 0.81 & -0.00 & 0.03 & 0 & 0 & 0 \\ -0.01 & 0.78 & 0.03 & 0 & 0 & 0 \\ 0.11 & 0.11 & 0.57 & 0 & 0 & 0 \\ -0.02 & 0.31 & -0.10 & 0 & 0 & 0 \\ -0.17 & 0.18 & 0.98 & 0 & 0 & 0 \\ 2.15 & 2.23 & -7.53 & 0 & 0 & 0 \end{bmatrix}, \ \Psi_{\theta=0.3} = \begin{bmatrix} 0.02 & -0.05 \\ -0.05 & 0.01 \\ 0.36 & 0.43 \\ -0.15 & 0.27 \\ -0.23 & 0.25 \\ 6.72 & 9.07 \end{bmatrix};$$

$$\Gamma_{\theta=.59} = \begin{bmatrix} 0.97 & -0.16 & 0.04 & 0 & 0 & 0 \\ -0.02 & 0.79 & 0.03 & 0 & 0 & 0 \\ 0.03 & 0.20 & 0.57 & 0 & 0 & 0 \\ -0.08 & 0.34 & -0.03 & 0 & 0 & 0 \\ -0.13 & 0.09 & 1.05 & 0 & 0 & 0 \\ 0.54 & 4.04 & -7.71 & 0 & 0 & 0 \end{bmatrix}, \ \Psi_{\theta=.59} = \begin{bmatrix} 0.25 & -0.24 \\ -0.04 & 0.01 \\ 0.31 & 0.54 \\ -0.12 & 0.10 \\ -0.21 & 0.08 \\ 5.65 & 11.20 \end{bmatrix}.$$

### A.4 Quality-augmented Capital

With the quality-augmented capital hypothesis, the set of state variables has one more element $\{\alpha_{t-1}, \beta_{t-1}, \overline{q}_t, k_t, A_t, B_t\}$ where $\overline{q}_t$ evolves according to (22); and the production functions become (23) and (24) where average quality augments capital services. The dynamic programming problem is again defined as in (A.1). We use the shorthands $\{c_t, d_t\}$ in (A.2) and (A.3), and define several new objects

$$\begin{aligned} DA_{t+1}(n_{at}) &= \frac{1}{\theta}\frac{\partial\left(a_{t+1}^{\theta}\alpha_{t+1}^{\theta}\right)}{\partial n_{at}} \\ &= \alpha_{t+1}^{\theta}a_{t+1}^{\theta-1}\frac{\mu Y_{at+1}}{\overline{q}_{t+1}}\left(S_k\nu\alpha_t^{\nu-1}\beta_t^{1-\nu}\right)\left(\xi n_{at}^{\xi-1}\right) \\ &\quad +a_{t+1}^{\theta}\alpha_{t+1}^{\theta-1}(1-\eta_\alpha)\left(\xi n_{at}^{\xi-1}\right), \end{aligned} \tag{A.92}$$

$$\begin{aligned} DB_{t+1}(n_{bt}) &= \frac{1}{\theta}\frac{\partial\left(b_{t+1}^{\theta}\beta_{t+1}^{\theta}\right)}{\partial n_{bt}} \\ &= \beta_{t+1}^{\theta}b_{t+1}^{\theta-1}\frac{\gamma Y_{bt+1}}{\overline{q}_{t+1}}\left(S_k(1-\nu)\alpha_t^{\nu}\beta_t^{-\nu}\right)\left(\xi n_{bt}^{\xi-1}\right) \\ &\quad +b_{t+1}^{\theta}\beta_{t+1}^{\theta-1}(1-\eta_\beta)\left(\xi n_{bt}^{\xi-1}\right). \end{aligned} \tag{A.93}$$

These expressions are related to the future marginal gains of current quality investment via capital augmentation. With these notations, the full system of FOC is

$$c_t^{-\sigma}d_t\alpha_t^{\theta}a_t^{\theta-1}(1-\mu)Y_{at}/l_{at} = \mu_{1t} \tag{A.94}$$

$$c_t^{-\sigma}d_t\beta_t^{\theta}b_t^{\theta-1}(1-\gamma)Y_{bt}/l_{bt} = \mu_{1t} \tag{A.95}$$

$$c_t^{-\sigma}d_t\alpha_t^{\theta}a_t^{\theta-1}\mu Y_{at}/k_{at} = \mu_{2t} \tag{A.96}$$

$$c_t^{-\sigma}d_t\beta_t^{\theta}b_t^{\theta-1}\gamma Y_{bt}/k_{bt} = \mu_{2t} \tag{A.97}$$

$$c_t^{-\sigma}d_t a_t^{\theta}\alpha_t^{\theta-1}\xi(n_{at})^{\xi-1} + \lambda E_t\left[c_{t+1}^{-\sigma}d_{t+1}DA_{t+1}(n_{at})\right] = \mu_{1t} \tag{A.98}$$

$$c_t^{-\sigma}d_t b_t^{\theta}\beta_t^{\theta-1}\xi(n_{bt})^{\xi-1} + \lambda E_t\left[c_{t+1}^{-\sigma}d_{t+1}DB_{t+1}(n_{bt})\right] = \mu_{1t} \tag{A.99}$$

$$\begin{gathered} c_t^{-\sigma}d_t\alpha_t^{\theta}a_t^{\theta-1} = \\ +\lambda E_t\left[c_{t+1}^{-\sigma}d_{t+1}\alpha_{t+1}^{\theta}a_{t+1}^{\theta-1}\mu(Y_{at+1}/k_{at+1})\nu\left(\phi x_{at}^{\nu-1}x_{bt}^{1-\nu}\right)\right] \end{gathered} \tag{A.100}$$

$$\begin{gathered} c_t^{-\sigma}d_t\beta_t^{\theta}b_t^{\theta-1} = \\ +\lambda E_t\left[c_{t+1}^{-\sigma}d_{t+1}\beta_{t+1}^{\theta}b_{t+1}^{\theta-1}\gamma(Y_{bt+1}/k_{bt+1})(1-\nu)\left(\phi x_{at}^{\nu}x_{bt}^{-\nu}\right)\right] \end{gathered} \tag{A.101}$$

$$l_{at}+l_{bt}+n_{at}+n_{bt} = 1;\ \mu_{1t} > 0 \tag{A.102}$$

$$k_{at}+k_{bt} = k_t;\ \mu_{2t} > 0. \tag{A.103}$$

The steady state of interest is $\{l_a, l_b, n_a, n_b, k, k_a, k_b, \alpha, \beta, \overline{q}, x_a, x_b\}$. To solve for these objects, we rely on (A.22)-(A.24) and the following 9 conditions

$$\alpha^{1/\xi}(1-\mu)Y_a/l_a = \{a[1+\lambda(1-\eta_\alpha)] + Y_a\lambda\mu S_k\nu\}\xi\eta_\alpha^{(1-1/\xi)} \tag{A.104}$$

$$\beta^{1/\xi}(1-\gamma)Y_b/l_b = \{b[1+\lambda(1-\eta_\beta)] + Y_b\lambda\gamma S_k(1-\nu)\}\xi\eta_\beta^{(1-1/\xi)} \tag{A.105}$$

$$\alpha^\theta a^{\theta-1}(1-\mu)(Y_a/l_a) = \beta^\theta b^{\theta-1}(1-\gamma)(Y_b/l_b) \tag{A.106}$$

$$\alpha^\theta a^{\theta-1}\mu(Y_a/k_a) = \beta^\theta b^{\theta-1}\gamma(Y_b/k_b) \tag{A.107}$$

$$1 = \lambda\mu(Y_a/k_a)\nu\left(\phi x_a^{\nu-1}x_b^{1-\nu}\right) \tag{A.108}$$

$$1 = \lambda\gamma(Y_b/k_b)(1-\nu)\left(\phi x_a^{\nu}x_b^{-\nu}\right) \tag{A.109}$$

$$\phi x_a^\nu x_b^{1-\nu} = \delta(k_a+k_b) \tag{A.110}$$

$$l_a + l_b = 1 - \left[(\eta_\alpha\alpha)^{1/\xi} + (\eta_\beta\beta)^{1/\xi}\right] \tag{A.111}$$

$$\overline{q} = \alpha^\nu\beta^{1-\nu}. \tag{A.112}$$

To approximate dynamic behavior around the steady state, we apply log-linearization and the method of undetermined coefficient once again. Let $z_t = [\widehat{\alpha}_t\ \widehat{\beta}_t\ \widehat{\overline{q}}_t\ \widehat{k}_{t+1}\ \widehat{l}_{bt}\ \widehat{k}_{bt}\ \widehat{x}_{bt}]'$ and $s_t = [\widehat{A}_t\ \widehat{\beta}_t]'$. The original system becomes

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t a_t^\theta\alpha_t^{\theta-1}\xi(n_{at})^{\xi-1} + \lambda c_{t+1}^{-\sigma}d_{t+1}DA_{t+1}(n_{at}) \\ -c_t^{-\sigma}d_t\alpha_t^\theta a_t^{\theta-1}(1-\mu)Y_{at}/l_{at}\end{array}\right\} = 0 \tag{A.113}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t b_t^\theta\beta_t^{\theta-1}\xi(n_{bt})^{\xi-1} + \lambda c_{t+1}^{-\sigma}d_{t+1}DB_{t+1}(n_{bt}) \\ -c_t^{-\sigma}d_t\beta_t^\theta b_t^{\theta-1}(1-\gamma)Y_{bt}/l_{bt}\end{array}\right\} = 0 \tag{A.114}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t\alpha_t^\theta a_t^{\theta-1}(1-\mu)Y_{at}/l_{at} \\ -c_t^{-\sigma}d_t\beta_t^\theta b_t^{\theta-1}(1-\gamma)Y_{bt}/l_{bt}\end{array}\right\} = 0 \tag{A.115}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t\alpha_t^\theta a_t^{\theta-1}\mu Y_{at}/k_{at} \\ -c_t^{-\sigma}d_t\beta_t^\theta b_t^{\theta-1}\gamma Y_{bt}/k_{bt}\end{array}\right\} = 0 \tag{A.116}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t\alpha_t^\theta a_t^{\theta-1} \\ -\lambda c_{t+1}^{-\sigma}d_{t+1}\alpha_{t+1}^\theta a_{t+1}^{\theta-1}\mu(Y_{at+1}/k_{at+1})\nu\left(\phi x_{at}^{\nu-1}x_{bt}^{1-\nu}\right)\end{array}\right\} = 0 \tag{A.117}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t\beta_t^\theta b_t^{\theta-1} \\ -\lambda c_{t+1}^{-\sigma}d_{t+1}\beta_{t+1}^\theta b_{t+1}^{\theta-1}\gamma(Y_{bt+1}/k_{bt+1})(1-\nu)\left(\phi x_{at}^{\nu}x_{bt}^{-\nu}\right)\end{array}\right\} = 0 \tag{A.118}$$

$$E_t\left\{\overline{q}_{t+1} - (1-S_k)\overline{q}_t - S_k\alpha_t^\nu\beta_t^{1-\nu}\right\} = 0. \tag{A.119}$$

## A.5 Technical Details of the US Application

Our main interest is the time path of US services-goods relative quality $\{Q\}_{t=1970}^{2006}$ which is driven by technology changes. To infer this unobservable time series, we need to find $\{\theta,\eta_\alpha,\eta_\beta,\xi\}$, $\{A_t,B_t\}_{t=1970}^{2006}$, and the initial state which make the model time paths $\{Y_{at},Y_{bt},p_t,S_{bt}\}_{t=1970}^{2006}$ have some moments close to those of data. The representative agent is assumed to perfectly observe and foresee the evolutions of $A_t$ and $B_t$, i.e. there are no uncertainties.

Let the model time length be $T+1$, which includes period 0 and the terminal period $T$. In $T$, there are no intertemporal benefits and hence no dynamic decisions. Thus we have the time line $[0,1,..,S,S+1,..,T]$, where 1 and $S$

respectively correspond the years 1970 and 2006, i.e. $S = 36$. Relabeling the time line, we need to find $\{A_t, B_t\}_{t=1}^T$. By normalization, in the year 2000, $A_{S-5} = 1$ and $B_{S-5} = 0.98$. The entire productivity time series are constructed based on the geometric mean growth rates of $\{A_t, B_t\}$ in the sample period 1970-2006. Next, $\{\alpha_0, \beta_0, k_0\}$ are assumed to be at the steady state values if the economy has productivity levels $\{A_1, B_1\}$ forever.

Given $\{A_t, B_t\}_{t=1}^T$ and $\{\alpha_0, \beta_0, k_0\}$, we need to find the four critical parameters. Our algorithm has two levels, the higher for finding these unknown parameters and the lower for having the corresponding equilibrium. At the higher level, we start with some guess on $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$; construct the differences between data and model moments; update the guess for the next round, and continue until the guess converges. At the lower level, given some $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$, we need to solve a large system of equations, which characterizes the corresponding equilibrium for $[1, .., T]$. The end point $T$ is chosen not very far from $S$ so that productivity projections are reliable; and at the same time, far enough so that the solutions in $[1, .., S]$ are not significantly influenced by terminal decisions. In fact, there is a trade-off between these two objectives.

Here are some details about the lower level. First, $\{\theta,\ \eta_\alpha,\ \eta_\beta,\ \xi\}$, $\{\alpha_0,\ \beta_0,\ k_0\}$, $\{A_t, B_t\}_{t=1}^T$ are given numbers. Second, all the dynamic choices in period $T$ are zero. Third, the state observed at the beginning of any period $t$ consists of $\{\alpha_{t-1}, \beta_{t-1}, k_t, A_t, B_t\}$, out of which the first three are endogenous. Fourth, as $\nu \in (0,1)$, $x_{at} > 0$ and $x_{bt} > 0$; and for $\xi \in (0,1)$, $n_{at} > 0$ and $n_{bt} > 0$. We choose the minimal set of equilibrium objects in each period to be $\{l_{bt}, k_{bt}, n_{at}, n_{bt}, x_{at}, x_{bt}\}$, out of which the last four govern the evolution of the state. Fifth, we have a system of $6T$ nonlinear equations derived from (A.5) to (A.14). Specifically, the system of necessary conditions for $t \in [1, T-1]$ is

$$c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} (1-\gamma) B_t k_{bt}^\gamma l_{bt}^{-\gamma} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} (1-\mu) A_t k_{at}^\mu l_{at}^{-\mu} = 0 \quad \text{(A.120)}$$

$$c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} \gamma B_t k_{bt}^{\gamma-1} l_{bt}^{1-\gamma} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} \mu A_t k_{at}^{\mu} l_{at}^{1-\mu} = 0 \quad \text{(A.121)}$$

$$\begin{aligned} &c_t^{-\sigma} d_t a_t^\theta \alpha_t^{\theta-1} \xi (n_{at})^{\xi-1} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} (1-\mu) A_t k_{at}^\mu l_{at}^{-\mu} \\ &+ \lambda c_{t+1}^{-\sigma} d_{t+1} a_{t+1}^\theta \alpha_{t+1}^{\theta-1} (1-\eta_\alpha) \xi (n_{at})^{\xi-1} = 0 \end{aligned} \quad \text{(A.122)}$$

$$\begin{aligned} &c_t^{-\sigma} d_t b_t^\theta \beta_t^{\theta-1} \xi (n_{bt})^{\xi-1} - c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} (1-\gamma) B_t k_{bt}^\gamma l_{bt}^{-\gamma} \\ &+ \lambda c_{t+1}^{-\sigma} d_{t+1} b_{t+1}^\theta \beta_{t+1}^{\theta-1} (1-\eta_\beta) \xi (n_{bt})^{\xi-1} = 0 \end{aligned} \quad \text{(A.123)}$$

$$\begin{aligned} &\lambda c_{t+1}^{-\sigma} d_{t+1} \alpha_{t+1}^\theta a_{t+1}^{\theta-1} \mu A_{t+1} k_{at+1}^{\mu-1} l_{at+1}^{1-\mu} \nu \phi x_{at}^{\nu-1} x_{bt}^{1-\nu} \\ &- c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} = 0 \end{aligned} \quad \text{(A.124)}$$

$$\begin{aligned} &\lambda c_{t+1}^{-\sigma} d_{t+1} \beta_{t+1}^\theta b_{t+1}^{\theta-1} \gamma B_{t+1} k_{bt+1}^{\gamma-1} l_{bt+1}^{1-\gamma} (1-\nu) \phi x_{at}^\nu x_{bt}^{-\nu} \\ &- c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} = 0. \end{aligned} \quad \text{(A.125)}$$

For the terminal period $T$, there are some changes as follows: two equations (A.120) and (A.121) still hold; (A.122) and (A.123) respectively become

$$c_t^{-\sigma} d_t a_t^\theta \alpha_t^{\theta-1} \xi (n_{at})^{\xi-1} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} (1-\mu) A_t k_{at}^\mu l_{at}^{-\mu} = 0 \quad \text{(A.126)}$$

$$c_t^{-\sigma} d_t b_t^{\theta} \beta_t^{\theta-1} \xi (n_{bt})^{\xi-1} - c_t^{-\sigma} d_t \beta_t^{\theta} b_t^{\theta-1} (1-\gamma) B_t k_{bt}^{\gamma} l_{bt}^{-\gamma} = 0; \quad (A.127)$$

and (A.124)-(A.125) are changed to

$$x_{at} = 0 \quad (A.128)$$

$$x_{bt} = 0. \quad (A.129)$$

Thus the system effectively has $6T-2$ nonlinear equations in $6T-2$ unknowns.

At the upper level, the objective function for moment matching is

$$F(\tau) = (m(\tau) - m)' (m(\tau) - m), \quad (A.130)$$

where $\tau = \{\theta, \eta_\alpha, \eta_\beta, \xi\}$; and $\{m, m(\tau)\}$ are vectors of data and model moments, respectively. Based on the limited data set, we only have two data moments: one is the angle between the linear trends of $B_t/A_t$ and $S_{bt}$, and one is the slope of $S_{bt}$ trend. It is noted that, as predicted by Proposition 3.3, the relative price does not bear information of these parameters, i.e. simulated $p_{bat}$ does not respond to changes in $\tau$. In this model, we are more interested in $\{\theta, \xi\}$ than $\{\eta_\alpha, \eta_\beta\}$. To overcome the under-identification problem, we need to fix $\{\eta_\alpha, \eta_\beta\}$. We first impose that $\eta_\alpha = \eta_\beta$ and do experiments on the discrete parameter space. Based on the objective function, we see that $\{\eta_\alpha, \eta_\beta\}$ should be small and cannot be zero. Next, by perturbing $\{\eta_\alpha, \eta_\beta\}$, we find $\eta_\beta > \eta_\alpha$ should hold. Finally, $\eta_\alpha = 0.0045$ and $\eta_\beta = 0.0055$. Conditional on $\{\eta_\alpha, \eta_\beta\}$, we employ Newton's method to search for $\{\theta, \xi\}$.

## A.6 A Model without Capital Accumulation

In this section, we will see that how relative productivity affects relative quality does not depend on whether there is capital accumulation in the model or not. We begin with the definition of the social planner problem. Without capital, the information sets become $\omega_t = \{A_t, B_t, \alpha_{t-1}, \beta_{t-1}\}$ and $\widehat{\omega}_t = \{A_t, B_t, \alpha_t, \beta_t\}$. The set of choice variables is $C_t = \{l_{at}, n_{at}, l_{bt}, n_{bt}\}$. The social planner solves

$$V(\omega_t) = \max_{C_t} \left\{ \begin{array}{c} \sum u(a_t, b_t; \alpha_t, \beta_t) + \lambda E_{\widehat{\omega}_t} V(\omega_{t+1}) \\ +\mu_t (1 - l_{at} - l_{bt} - n_{at} - n_{bt}) \end{array} \right\}$$

$s.t.$

$$\begin{aligned} \alpha_t &= (1-\eta_\alpha)\alpha_{t-1} + (n_{at})^{\xi} \\ \beta_t &= (1-\eta_\beta)\beta_{t-1} + (n_{bt})^{\xi} \\ a_t &= A_t l_{at} \\ b_t &= B_t l_{bt}. \end{aligned} \quad (A.131)$$

Define $c_t = [(\alpha_t a_t)^\theta + (\beta_t b_t)^\theta]^{1/\theta}$. The necessary and sufficient conditions are

$$c_t^{1-\sigma-\theta} \alpha_t^{\theta} a_t^{\theta-1} A_t = \mu_t \quad (A.132)$$

$$c_t^{1-\sigma-\theta}\beta_t^{\theta}b_t^{\theta-1}B_t = \mu_t \tag{A.133}$$

$$\begin{aligned} & c_t^{1-\sigma-\theta}a_t^{\theta}\alpha_t^{\theta-1}\xi\left(n_{at}\right)^{\xi-1} \\ +\lambda E_t & \left[c_{t+1}^{1-\sigma-\theta}a_{t+1}^{\theta}\alpha_{t+1}^{\theta-1}\left(1-\eta_\alpha\right)\xi\left(n_{at}\right)^{\xi-1}\right] = \mu_t \end{aligned} \tag{A.134}$$

$$\begin{aligned} & c_t^{1-\sigma-\theta}b_t^{\theta}\beta_t^{\theta-1}\xi\left(n_{bt}\right)^{\xi-1} \\ +\lambda E_t & \left[c_{t+1}^{1-\sigma-\theta}b_{t+1}^{\theta}\beta_{t+1}^{\theta-1}\left(1-\eta_\beta\right)\xi\left(n_{bt}\right)^{\xi-1}\right] = \mu_t \end{aligned} \tag{A.135}$$

$$l_{at} + l_{bt} + n_{at} + n_{bt} = 1;\ \mu_t > 0. \tag{A.136}$$

In steady state, we have 8 endogenous objects of interest $\alpha$, $\beta$, $a$, $b$, $l_a$, $l_b$, $n_a$, and $n_b$. Especially, we need to solve $\beta/\alpha$ in terms of $A$ and $B$ as well as the parameters. After some manipulations, the system of FOC is reduced to

$$\begin{aligned} \alpha^{\theta}a^{\theta-1}A &= \beta^{\theta}b^{\theta-1}B \\ \alpha A &= a\left[1+\lambda\left(1-\eta_\alpha\right)\right]\xi\left(n_a\right)^{\xi-1} \\ \beta B &= b\left[1+\lambda\left(1-\eta_\beta\right)\right]\xi\left(n_b\right)^{\xi-1} \\ l_a + l_b + n_a + n_b &= 1 \\ a &= Al_a \\ b &= Bl_b \\ \alpha\eta_\alpha &= \left(n_a\right)^{\xi} \\ \beta\eta_\beta &= \left(n_b\right)^{\xi}. \end{aligned}$$

Eventually, it is not difficult to arrive at the quality ratio as follows

$$\begin{aligned} \frac{\beta}{\alpha} &= \frac{\eta_\alpha}{\eta_\beta}\frac{\left\{\left(\eta_\beta^{\theta}\left(\left[1+\lambda\left(1-\eta_\alpha\right)\right]\eta_\alpha\right)^{\theta\xi}A^{\theta}\right)^{1/(\theta\xi+\theta-1)}\left[1+\lambda\left(1-\eta_\beta\right)\right]\xi\eta_\beta\right\}^{\xi}}{\left\{\left(\eta_\alpha^{\theta}\left(\left[1+\lambda\left(1-\eta_\beta\right)\right]\eta_\beta\right)^{\theta\xi}B^{\theta}\right)^{1/(\theta\xi+\theta-1)}\left[1+\lambda\left(1-\eta_\alpha\right)\right]\xi\eta_\alpha\right\}^{\xi}} \\ \frac{\beta}{\alpha} &= C_7\left(\frac{B}{A}\right)^{\theta\xi/(1-\theta\xi-\theta)}. \end{aligned} \tag{A.137}$$

As in the baseline model, we have

$$sign[\partial(\beta/\alpha)/\partial(B/A)] = sign[-\theta(\theta\xi+\theta-1)]. \tag{A.138}$$